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ERRATUM TO "THE HOMOGENEOUS COORDINATE RING OF A TORIC VARIETY"

DAVID A. COX

My paper "The Homogeneous Coordinate Ring of a Toric Variety" [*J. Algebraic Geometry* **4** (1995), 17–50] has some incorrect statements before and during the proof of Proposition 4.3. The purpose of this note is to correct these errors and give a valid proof of the proposition. I am very grateful to Alexander Duncan for bringing this matter to my attention.

We will use the same notation as the paper, hereinafter referred to as [2]. The major error in the paper occurs in the discussion following the statement of Theorem 4.2 of [2], where we assert that the set $\mathrm{End}_g(S)$ of graded $\mathbb{C}$-algebra homomorphisms $\phi : S \to S$ with $\phi(1) = 1$ is a $\mathbb{C}$-algebra. Here, S is the homogeneous coordinate ring of the toric variety X. Nowadays S is called the *total coordinate ring* (see [3]) or the *Cox ring* (see [5]). However, as pointed out to me by Duncan, the composition of two $\mathbb{C}$-algebra homomorphisms is again a $\mathbb{C}$-algebra homomorphism, but the same is not true for their sum. So right now, the best we can say is that $\mathrm{End}_g(S)$ is a monoid under composition.

The proof of Proposition 4.3 in [2] was based on the faulty assumption that $\mathrm{End}_g(S)$ is a $\mathbb{C}$-algebra. Hence the main task of this note is to give a correct proof of the proposition. Here is the proposition, with some improvements suggested by the referee.

Proposition 4.3. *Let X be a complete toric variety, and let S be its homogeneous coordinate ring. Then*

(i) $\mathrm{Aut}_g(S)$ *is a connected affine algebraic group of dimension equal to* $\sum_{i=1}^{s} |\Delta_i| \dim_{\mathbb{C}} S_{\alpha_i}$, *and* $(\mathbb{C}^*)^{\Delta(1)} \subset \mathrm{Aut}_g(S)$ *is a maximal torus.*
(ii) *The unipotent radical* R_u *of* $\mathrm{Aut}_g(S)$ *is isomorphic as a variety to an affine space of dimension* $\sum_{i=1}^{s} |\Delta_i|(\dim_{\mathbb{C}} S_{\alpha_i} - |\Delta_i|)$.
(iii) $\mathrm{Aut}_g(S)$ *has a closed subgroup* G_s *isomorphic to the reductive group* $\prod_{i=1}^{s} \mathrm{GL}(S'_{\alpha_i})$ *of dimension* $\sum_{i=1}^{s} |\Delta_i|^2$. *Also,* $(\mathbb{C}^*)^{\Delta(1)} \subseteq G_s$.
(iv) $\mathrm{Aut}_g(S)$ *is isomorphic to the semidirect product* $R_u \rtimes G_s$.

Proof. To simplify notation, we write the direct sum decompostion $S_{\alpha_i} = S'_{\alpha_i} \oplus S''_{\alpha_i}$ from (7) of [2] as $S_i = S'_i \oplus S''_i$. Since elements of $\mathrm{End}_g(S)$ are $\mathbb{C}$-linear, preserve degrees, and are determined uniquely by their values on the variables x_ρ, we have a bijection of sets

$$\mathrm{End}_g(S) \simeq \prod_{i=1}^{s} \mathrm{Hom}_{\mathbb{C}}(S'_i, S_i), \tag{E1}$$

and we also have an injection

$$\mathrm{End}_g(S) \hookrightarrow \prod_{i=1}^{s} \mathrm{End}_{\mathbb{C}}(S_i) \tag{E2}$$

that is compatible with composition.

We first show that $\mathrm{Im}(\mathrm{End}_g(S)) \subset \prod_{i=1}^{s} \mathrm{End}_{\mathbb{C}}(S_i)$ is a variety. Recall from [2] that $\phi(S''_i) \subset S''_i$. It follows that $\phi \in \mathrm{End}_{\mathbb{C}}(S)$ corresponds via (E2) to a collection of matrices

$$\begin{pmatrix} A_i & 0 \\ B_i & C_i \end{pmatrix} \in \mathrm{End}_{\mathbb{C}}(S_i), \quad i = 1, \ldots, s, \tag{E3}$$

where we use the canonical basis of $S_i = S'_i \oplus S''_i$ given by monomials of degree α_i to identify matrices with linear maps. Note that the S'_i-columns $\left(\begin{smallmatrix} A_i \\ B_i \end{smallmatrix}\right)$ of (E3) are the data that make up the map (E1). The matrices C_i come from evaluating ϕ at monomials in S''_i, which are products of ≥ 2 variables that lie in various S_j for $j \neq i$ (this follows from $S_0 = \mathbb{C}$). Hence the entries in C_i are detemined by the matrices A_j, B_j for $j \neq i$. We will say more about this below.

One fact we will need is how (E1) relates to composition. Suppose that $\phi, \psi \in \mathrm{End}_g(S)$ maps to matrices

$$\begin{pmatrix} A_i & 0 \\ B_i & C_i \end{pmatrix}, \begin{pmatrix} A'_i & 0 \\ B'_i & C'_i \end{pmatrix}, \quad i = 1, \ldots, s.$$

Since (E2) is compatible with composition, we see that $\phi \circ \psi$ corresponds to the products

$$\begin{pmatrix} A_i & 0 \\ B_i & C_i \end{pmatrix} \begin{pmatrix} A'_i & 0 \\ B'_i & C'_i \end{pmatrix} = \begin{pmatrix} A_i A'_i & 0 \\ B_i A'_i + C_i B'_i & C_i C'_i \end{pmatrix}, \quad i = 1, \ldots, s.$$

It follows that in the bijection (E1), we have

$$\text{if } \phi \longleftrightarrow \begin{pmatrix} A_i \\ B_i \end{pmatrix} \text{ and } \psi \longleftrightarrow \begin{pmatrix} A'_i \\ B'_i \end{pmatrix}, \text{ then } \phi \circ \psi \longleftrightarrow \begin{pmatrix} A_i A'_i \\ B_i A'_i + C_i B'_i \end{pmatrix}. \tag{E4}$$

This will be useful later in the proof.

The next step is to write down the equations that define $\mathrm{End}_g(S)$ inside of $\prod_{i=1} \mathrm{End}_{\mathbb{C}}(S_i)$ in (E2). Our treatment is inspired by [1, Prop. 5.12]. The equations come from two sources:

- First, all of the entries in the upper right-hand block must be zero. This is the "0" in (E3).
- Second, suppose that we have monomials $x^D, x^E \in S_i''$. Given $\phi \in \mathrm{End}_g(S)$, we have

$$\phi(x^D) = \cdots + c^i_{ED} x^E + \cdots ,$$

where c^i_{ED} is the corresponding entry in C_i in (E3) for ϕ. But x^D is a product of variables $x_{\rho_1} \cdots x_{\rho_\ell}$, where we allow duplications. Note that $x_{\rho_j} \notin S'_{\alpha_i}$ since $S_0 = \mathbb{C}$. It follows that

$$\begin{aligned} c^i_{ED} &= \text{coefficient of } x^E \text{ in } \phi(x^D) \\ &= \text{coefficient of } x^E \text{ in } \phi(x_{\rho_1}) \cdots \phi(x_{\rho_\ell}). \end{aligned}$$

Each $\phi(x_{\rho_j})$ is a linear combination of monomials whose coefficients are the corresponding entries in the matrices A_{k_j}, B_{k_j}, where $x_{\rho_j} \in S_{k_j}$, i.e., $\deg(x_{\rho_j}) = \alpha_{k_j}$. Hence we get an equation linking c^i_{ED} with entries in A_{k_j}, B_{k_j}, $j = 1, \ldots, \ell$.

This analysis shows that $\mathrm{End}_g(S)$ is a linear algebraic monoid in the sense of [6]. Since $\mathrm{Aut}_g(S)$ is the group of invertible elements of $\mathrm{End}_g(S)$, it follows from [6] that $\mathrm{Aut}_g(S)$ is an algebraic group.

We will need the following characterization of which elements of $\mathrm{End}_g(S)$ are invertible: if $\phi \in \mathrm{End}_g(S)$ corresponds to matrices (E3), then

$$\text{(E5)} \qquad \phi \in \mathrm{Aut}_g(S) \iff A_i, C_i \text{ are invertible for } i = 1, \ldots, s.$$

One direction is obvious. For the other, suppose that the A_i, C_i are all invertible. Then consider the element $\psi \in \mathrm{End}_g(S)$ such that

$$\psi \longleftrightarrow \begin{pmatrix} A_i^{-1} \\ -C_i^{-1} B_i A_i^{-1} \end{pmatrix}$$

via (E1). Using (E4), one obtains $\phi \circ \psi \longleftrightarrow \left(\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right)$, so that $\phi \circ \psi$ is the identity. But then the matrices associated to ϕ and ψ multiply to the identity in each $\mathrm{End}_{\mathbb{C}}(S_i)$, which means that the same is true when we reverse the order. Hence $\psi \circ \phi$ is also the identity, which proves that $\phi \in \mathrm{Aut}_g(S)$. This completes the proof of (E5).

As in [2], let

$$\mathcal{N} = \prod_{i=1}^{s} \mathrm{Hom}_{\mathbb{C}}(S_i', S_i'').$$

To define $1 + \mathcal{N} \subset \mathrm{End}_g(S)$, we have to be careful since endomorphisms cannot be added. We let $1 + \mathcal{N}$ consist of all $\phi \longleftrightarrow \left(\begin{smallmatrix} I \\ B_i \end{smallmatrix}\right)$ via (E1), where $B_i \in \mathrm{Hom}_{\mathbb{C}}(S_i', S_i'')$. Then an element $\phi \in 1 + \mathcal{N}$ gives matrices

$$\text{(E6)} \qquad \begin{pmatrix} I & 0 \\ B_i & C_i \end{pmatrix} \in \mathrm{End}_{\mathbb{C}}(S_i) \quad i = 1, \ldots, s.$$

We claim that these matrices are all unipotent.

To study C_i, we order the monomials in S_i'' so that x^D appears before x^E whenever the total degree of x^D (as a monomial in the polynomial ring S) is strictly smaller than the total degree of x^E. Take $x^D \in S_i$ and write $x^D = x_{\rho_1} \cdots x_{\rho_l}$, so that x^D has total degree ℓ. Applying ϕ, we get

$$\phi(x_\rho) = \prod_{j=1}^{\ell} \phi(x_{\rho_j}) = \prod_{j=1}^{\ell} \left(x_{\rho_j} + \textstyle\sum_{x^E \in S''_{\deg(\mathrm{x}_{\rho_\mathrm{j}})}} b_{E,j} x^E\right).$$

Since every monomial in $S''_{\deg(\mathrm{x}_{\rho_\mathrm{j}})}$ has total degree at least two, multiplying out the last product gives

$$\phi(x^D) = x^D + \text{terms of higher total degree}.$$

Given how the monomials in S''_{α_i} are ordered, it follows that C_i is lower triangular with 1's on the main diagonal. Then the same is true for (E6), so that (E6) is unipotent as claimed.

Now that we know that C_i is invertible, (E5) and (E6) imply that ϕ is invertible. Hence we have proved that $1 + \mathcal{N} \subset \mathrm{Aut}_g(S)$. Notice also that $1 + \mathcal{N}$ is a closed subgroup of $\mathrm{Aut}_g(S)$ by (E4) and (E6).

Now we get to our main task, which is to establish the exact sequence of groups

$$\text{(E7)} \qquad 1 \longrightarrow 1 + \mathcal{N} \xrightarrow{\alpha} \mathrm{Aut}_g(S) \xrightarrow{\beta} \prod_{i=1}^{s} \mathrm{GL}(S_i') \longrightarrow 1.$$

This is the exact sequence (9) of [2].

The map α is the inclusion $1 + \mathcal{N} \subset \mathrm{Aut}_g(S)$ proved above. The map β is also easy to describe: if $\phi \in \mathrm{Aut}_g(S)$ is specified by $\left(\begin{smallmatrix} A_i \\ B_i \end{smallmatrix}\right)$, then the A_i are invertible by (E5) and hence give an element of $\prod_{i=1}^{s} \mathrm{GL}(S_i')$. This is $\beta(\phi)$. Note that β is a group homomorphism by (E4).

The map α is clearly injective, and (E7) is exact at $\mathrm{Aut}_g(S)$ by the definition of $1+\mathcal{N}$. It remains to prove that β is onto. Suppose that we have invertible matrices $A_i \in \mathrm{GL}(S_i')$ for $i = 1, \ldots, s$. Then consider $\phi, \psi \in \mathrm{End}_g(S)$ such that

$$\text{(E8)} \qquad \phi \longleftrightarrow \begin{pmatrix} A_i \\ 0 \end{pmatrix} \text{ and } \psi \longleftrightarrow \begin{pmatrix} A_i^{-1} \\ 0 \end{pmatrix}$$

via (E1). Using (E4), one computes that

$$\phi \circ \psi \longleftrightarrow \begin{pmatrix} A_i A_i^{-1} \\ 0 \cdot A_i^{-1} + C_i \cdot 0 \end{pmatrix} = \begin{pmatrix} I \\ 0 \end{pmatrix}.$$

This proves that $\phi \circ \psi$ is the identity, and switching ϕ and ψ shows that $\psi \circ \phi$ is the identity as well. Thus $\phi \in \mathrm{Aut}_g(S)$. Since β maps ϕ to the A_i, surjectivity follows.

Hence (E7) is exact, and we also know that $1 + \mathcal{N}$ is unipotent. Then part (ii) of the proposition follows because, as a variety, we have $1 + \mathcal{N} \simeq \mathcal{N}$, which is an affine space of the required dimension.

For part (iii), note that the first half of (E8) gives a section

$$s^* : \prod_{i=1}^{s} \mathrm{GL}(S_i') \longrightarrow \mathrm{Aut}_g(S)$$

of the exact sequence (E7). Note that s^* is a group homomorphism by (E4). The image is easily seen to be an algebraic subgroup containing $(\mathbb{C}^*)^{\Delta(1)}$. This proves part (iii) of the proposition, and parts (iv) and (i) now follow without difficulty in view of (E7). The proof is complete. □

Here are some final comments:

- Lemma 1.3 of [2] is only used in the invalid proof of Proposition 4.3 in [2]. Hence this lemma can be ignored when reading the paper. In the proof of Proposition 4.3 presented here, the ordering of Lemma 1.3 is replaced by the total degree ordering on the polynomial ring S.
- The sentence following the first display in the proof of Proposition 4.5 of [2] needs to be modified: "is the maximal torus and hence lies in G_s" should be "is a maximal torus contained in G_s."
- In [4] Demazure gives a functorial construction of the automorphism group of a toric variety X. In [2] the approach is more concrete, based on the construction of $\mathrm{Aut}_g(S)$ as a matrix group. It would be useful to show that these two methods lead to the same algebraic group.

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References

[1] Winfried Bruns and Joseph Gubeladze, *Polytopes, rings, and K-theory*, Springer Monographs in Mathematics, Springer, Dordrecht, 2009. MR2508056 (2010d:19001)

[2] David A. Cox, *The homogeneous coordinate ring of a toric variety*, J. Algebraic Geom. **4** (1995), no. 1, 17–50. MR1299003 (95i:14046)

[3] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. MR2810322 (2012g:14094)

[4] Michel Demazure, *Sous-groupes algébriques de rang maximum du groupe de Cremona*, Ann. Sci. École Norm. Sup. (4) **3** (1970), 507–588 (French). MR0284446 (44 #1672)

[5] Yi Hu and Sean Keel, *Mori dream spaces and GIT*, Michigan Math. J. **48** (2000), 331–348, DOI 10.1307/mmj/1030132722. Dedicated to William Fulton on the occasion of his 60th birthday. MR1786494 (2001i:14059)

[6] Mohan S. Putcha, *Linear algebraic monoids*, London Mathematical Society Lecture Note Series, vol. 133, Cambridge University Press, Cambridge, 1988. MR964690 (90a:20003)

Department of Mathematics, Amherst College, Amherst, Massachusetts 01002
E-mail address: dac@math.amherst.edu