

Article

Mirror Symmetry and Polar Duality of Polytopes

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Abstract: This expository article explores the connection between the polar duality from polyhedral geometry and mirror symmetry from mathematical physics and algebraic geometry. Topics discussed include duality of polytopes and cones as well as the famous quintic threefold and the toric variety of a reflexive polytope.

Keywords: polar duality; reflexive polytope; mirror symmetry

1. Introduction

In 1991, a group of physicists astounded the algebraic geometry community with conjectures for the number of rational curves of degree d lying on the famous *quintic threefold*, a variety of dimension three and degree five in 4-dimensional projective space. The key tool used in the paper [1] is *mirror symmetry*, which is an isomorphism of quantum field theories that turns a difficult counting problem on the quintic threefold into a much simpler problem on the so-called *quintic mirror*.

All of this seems far removed from the polar duality of polytopes, but as we will see in this article, there is a surprisingly deep connection based on the *reflexive polytopes* introduced by Batyrev [2] in the context of toric geometry.

2. Reflexive Polytopes

A *polytope* $P \subseteq \mathbb{R}^d$ is the convex hull of finitely many points in \mathbb{R}^d . If $0 \in \text{Int}(P)$, then its *polar or dual* is

$$P^\circ = \{u \in \mathbb{R}^d \mid u \cdot m \geq -1 \text{ for all } m \in P\}.$$

It is well-known that P° is a polytope with $0 \in \text{Int}(P^\circ)$ and $(P^\circ)^\circ = P$ (see, for instance, ([3] 2.3)).

Figure 1 shows a classic example of a polytope and its dual in three dimensions. The polytopes relevant to mirror symmetry have dimension four. In Example 2.1, we give the polytopes that underlie the quintic threefold and its mirror.

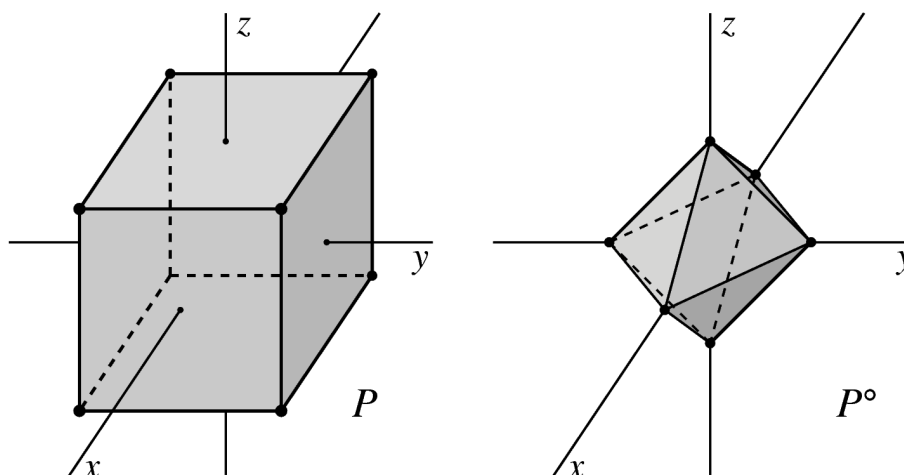


Figure 1. A cube $P \subseteq \mathbb{R}^3$ and its dual octahedron P° . Reprinted from [4] (p. 81) with permission of the American Mathematical Society.

Example 2.1. Consider the *Standard 4-Simplex*

$$\Delta_4 = \text{Conv}(0, e_1, e_2, e_3, e_4) = \{(m_1, m_2, m_3, m_4) \in \mathbb{R}^4 \mid m_i \geq 0, m_1 + m_2 + m_3 + m_4 \leq 1\},$$

where e_1, e_2, e_3, e_4 are the standard basis of \mathbb{R}^4 and “Conv” denotes convex hull. Then 0 is an interior point of the polytope

$$P = 5\Delta_4 - (1, 1, 1, 1),$$

and the dual of P is

$$P^\circ = \text{Conv}(e_1, e_2, e_3, e_4, (-1, -1, -1, -1)).$$

As polytopes, P and P° are simplices. But in terms of *lattice points* (points with integer coordinates, i.e., $\mathbb{Z}^4 \subseteq \mathbb{R}^4$), there is a substantial difference:

- P has 125 lattice points, i.e., $|P \cap \mathbb{Z}^4| = 125$.
- P° has 6 lattice points, i.e., $|P^\circ \cap \mathbb{Z}^4| = 6$.

We will see later that P and P° give rise to mirror manifolds where a hard problem on P transforms into a simpler problem on P° because of the small number of lattice points.

This example suggests that lattice points have an important role to play. In general, a *lattice polytope* or *integer polytope* is the convex hull of a finite subset of $\mathbb{Z}^d \subseteq \mathbb{R}^d$. The book [5] gives a nice introduction to lattice polytopes and their associated counting problems.

The cube P in Figure 1 becomes a lattice polytope when we choose coordinates so that its vertices are $\pm e_1 \pm e_2 \pm e_3$ for the standard basis e_1, e_2, e_3 of \mathbb{R}^3 . Then P° is the lattice polytope with vertices $\pm e_1, \pm e_2, \pm e_3$.

In general, polar duality interacts poorly with lattice polytopes—what happened in Figure 1 is rather special. For example, if we double the size of the cube to get $2P$, then its dual is

$$(2P)^\circ = \frac{1}{2}P^\circ,$$

which fails to be a lattice polytope. This leads to the following definition, due to Batyrev [2].

Definition 2.1. A lattice polytope P is *reflexive* if $0 \in \text{Int}(P)$ and P° is also a lattice polytope.

Figure 2 shows a less obvious example of a reflexive polytope.

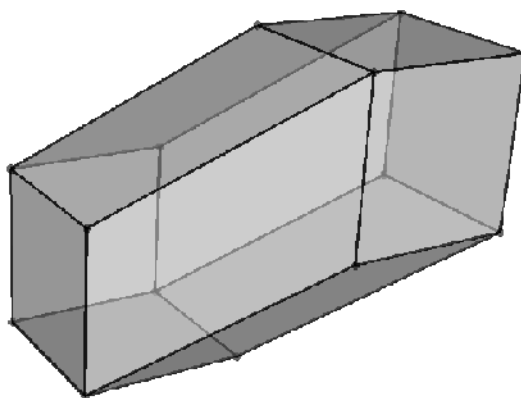


Figure 2. A reflexive polytope in \mathbb{R}^3 with 14 vertices. Reprinted from [6] (p. 180) with permission of the American Mathematical Society.

If P is reflexive, then it is straightforward to show that 0 is the unique interior lattice point of P . For another way to see what reflexive means, we first recall another description of the dual polytope. Given an arbitrary polytope with $0 \in \text{Int}(P)$, every facet F of P has a unique inward-pointing facet normal u_F with the property that

$$F = \{m \in P \mid m \cdot u_F = -1\}.$$

Then one can show that

$$P^\circ = \text{Conv}(u_F \mid F \text{ is a facet of } P).$$

Hence, a lattice polytope P with $0 \in \text{Int}(P)$ is reflexive if and only if $u_F \in \mathbb{Z}^d$ for all facets F of P .

Classifying reflexive polytopes in \mathbb{R}^d is an important problem because of their relevance to mirror symmetry. By “classify”, we mean up to a lattice equivalence of \mathbb{Z}^d , *i.e.*, up to coordinate change by an element of $\text{GL}(d, \mathbb{Z})$. In dimension two, the 16 classes of reflexive polygons are shown in Figure 3. It is a fun exercise to match polygons with their duals. In some cases, one needs to change coordinates by an element of $\text{GL}(2, \mathbb{Z})$ to identify the dual. Some of the polygons are self-dual up to $\text{GL}(2, \mathbb{Z})$.

In dimension three, there are 4319 classes of reflexive polytopes, and such number balloons to 473,800,776 in dimension four, an impressive calculation done by Kreuzer and Skarke [7] in 2002. As we will soon see, 4-dimensional reflexive polytopes are important in mirror symmetry.

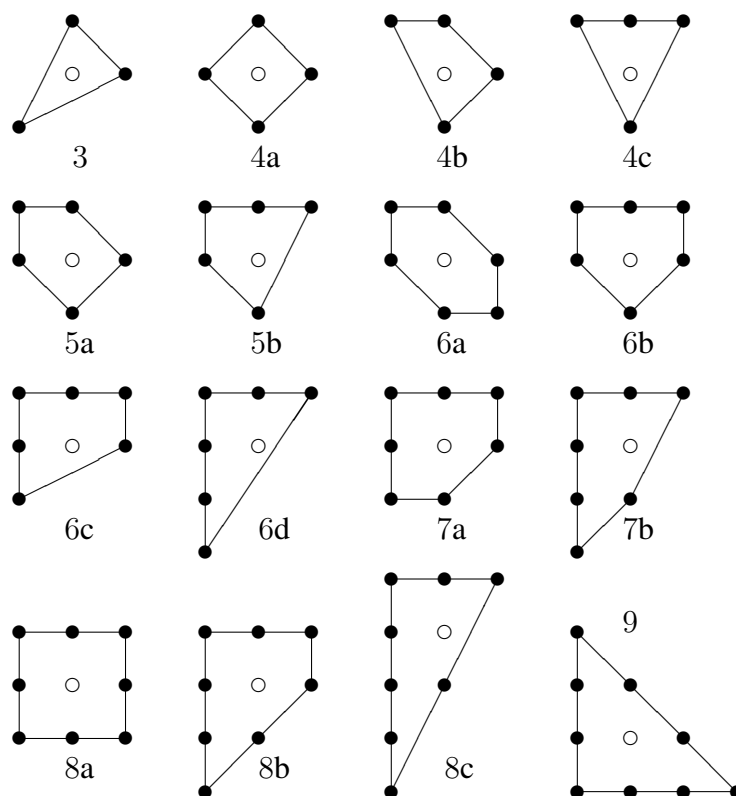


Figure 3. The 16 classes of reflexive lattice polygons in \mathbb{R}^2 . The open circles represent the origin and the labels record the number of boundary lattice points. Reprinted from [4] (p. 382) with permission of the American Mathematical Society.

3. Mirror Symmetry

String theory from mathematical physics is based on a 10-dimensional universe, where four dimensions are the familiar space-time of general relativity and the remaining six dimensions are where the quantum effects take place. *The Elegant Universe* by Greene [8] describes this model of the universe for a general audience.

The 6-dimensional quantum piece is a (very small) compact manifold, about the size of Planck's constant. To make this manifold support the kind of quantum field theory required by string theory, the manifold needs to have a complex structure with a trivial canonical bundle and vanishing first Betti number. Six real dimensions mean three complex dimensions, and the complex manifolds that arise are called *Calabi–Yau threefolds*. We recommend *The Shape of Inner Space* by Yau and Nadis [9] for a non-technical account of these spaces.

Example 3.1. The simplest Calabi–Yau threefold is the *quintic threefold*. We start with \mathbb{P}^4 , the 4-dimensional projective space over the complex numbers. Points in \mathbb{P}^4 have *homogeneous coordinates* $(x_0, x_1, x_2, x_3, x_4)$, where the coordinates never vanish simultaneously and two sets of coordinates give the same point if and only if they differ by a nonzero scalar multiple. A homogeneous equation

$$F(x_0, x_1, x_2, x_3, x_4) = 0, \quad \deg(F) = 5,$$

defines the *quintic threefold* $V \subseteq \mathbb{P}^4$. For most choices of F , V is smooth, *i.e.*, is a manifold. Then having degree five guarantees that V is a Calabi–Yau threefold.

Mirror symmetry involves some sophisticated physics and algebraic geometry. The string theories used in mirror symmetry are superconformal field theories (SCFTs) with twisted versions called the *A-model* and the *B-model*. The SCFT on a Calabi–Yau threefold V depends on two types of parameters:

- *Kähler moduli parameters* that encode the metric on V and control the A-model.
- *Complex moduli parameters* that encode the complex structure of V and control the B-model.

The number of parameters of each type is determined by the *Hodge numbers* of V , defined by

$$h^{pq}(V) = \dim H^q(V, \Omega_V^p). \quad (1)$$

In particular, $h^{11}(V)$ is the number of Kähler parameters and $h^{21}(V)$ is the number of complex parameters.

The central idea of mirror symmetry is that given a family of Calabi–Yau threefolds V , there should be a *mirror family* of Calabi–Yau threefolds V° such that the corresponding SCFTs are isomorphic in a way that *interchanges the A- and B-models and hence interchanges Kähler and complex moduli*. In particular, V and V° satisfy

$$h^{11}(V^\circ) = h^{21}(V) \quad \text{and} \quad h^{21}(V^\circ) = h^{11}(V). \quad (2)$$

The first explicit example of mirror symmetry involves the quintic threefold $V \subseteq \mathbb{P}^4$ and its mirror, which is defined as follows.

Example 3.2. To construct the *quintic mirror*, we begin with the threefold in $W \subseteq \mathbb{P}^4$ defined by

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 + \psi x_0 x_1 x_2 x_3 x_4 = 0, \quad \psi \in \mathbb{C} \setminus \{0\}. \quad (3)$$

Let $\zeta = e^{2\pi i/5}$ and note that the group

$$G = \{(\zeta^{a_0}, \zeta^{a_1}, \zeta^{a_2}, \zeta^{a_3}, \zeta^{a_4}) \mid a_0 + a_1 + a_2 + a_3 + a_4 \equiv 0 \pmod{5}\}$$

acts on W . The quotient W/G is not smooth, so one needs to do a *resolution of singularities* to produce a smooth variety. If done carefully, the result is Calabi–Yau. This is the quintic mirror V° .

The single parameter ψ in the above equation gives the complex moduli parameter $\lambda = \psi^5$, so that $h^{21}(V^\circ) = 1$. This is consistent with (2) since the quintic threefold V has $h^{11}(V) = 1$.

As explained in the 1991 paper [1], we have the following miracle:

- Rational curves on V of various degrees are important in *enumerative algebraic geometry*. These can be encoded into *Gromov–Witten invariants* that are intimately related to Kähler moduli and the A-model of V .
- By mirror symmetry and the *mirror map*, we can switch to the B-model of V° , where the complex moduli and B-model can be studied by the differential equations that arise in the *variation of Hodge structure* on V° . This is straightforward to study since $h^{21}(V^\circ) = 1$.

- The result is an explicit formula for *all* of the Gromov–Witten invariants! A careful description of the formula is appears in ([10] Chapter 2).

This formula astounded the algebraic geometers: a far-out theory from physics turns a hard problem in enumerative algebraic geometry into a solvable problem in Hodge theory. Mind blowing!

Since SCFTs are not mathematically rigorous, the formulas in [1] were regarded as conjectures by the mathematics community. These formulas are now theorems (see ([10] Chapter 11) for proofs and references to the original papers), though the full story of the relation between rational curves on the quintic threefold and Gromov–Witten invariants is still not fully understood.

4. Mirror Symmetry and Reflexive Polytopes

After the quintic threefold, many other examples of mirror manifolds were discovered using *weighted projective spaces*. However, there were some examples where the mirrors seemed to be missing. This was rectified in 1994 when Batyrev [2] pointed out that a 4-dimensional reflexive polytope gives a pair of Calabi–Yau threefolds that are natural candidates for a mirror pair.

An elementary approach to the transition from 4-dimensional reflexive polytopes to Calabi–Yau threefolds can be found in the paper [11], which is accessible to undergraduates. A more sophisticated approach uses the theory of *toric varieties*, which is an important part of modern algebraic geometry. A brief introduction to toric varieties can be found in [12], while [4] gives a comprehensive treatment.

For us, the starting point is that a lattice polytope $P \subseteq \mathbb{R}^d$ gives a d -dimensional toric variety X_P as follows. Let t_1, \dots, t_d be coordinates on the *torus* $(\mathbb{C}^*)^d$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Then a lattice point $m = (a_1, \dots, a_d) \in \mathbb{Z}^d$ gives the *Laurent monomial*

$$t^m = t_1^{a_1} \cdots t_d^{a_d}.$$

Negative exponents are allowed, but this is fine since $t_i \in \mathbb{C}^*$. If P has “enough lattice points” (this can be made precise) and $P \cap \mathbb{Z}^d = \{m_0, \dots, m_s\}$, then the map

$$t = (t_1, \dots, t_d) \in (\mathbb{C}^*)^d \mapsto (t^{m_0}, \dots, t^{m_s}) \in \mathbb{P}^s$$

is injective, and the toric variety $X_P \subseteq \mathbb{P}^s$ is defined to be the closure of the image. Note that X_P contains the torus $(\mathbb{C}^*)^d$, hence the name “toric variety”. (When P does not have enough lattice points, one uses nP for $n \gg 0$ in \mathbb{Z} —see [4] §2.2 and §2.3.)

Besides helping to define X_P , the lattice points of P also give some interesting hypersurfaces of X_P . Namely, if $P \cap \mathbb{Z}^d = \{m_0, \dots, m_s\}$ as above, then the equation

$$c_0 t^{m_0} + \cdots + c_s t^{m_s} = 0 \tag{4}$$

defines a hypersurface in $(\mathbb{C}^*)^d$. Its closure in X_P is the desired hypersurface $V \subseteq X_P$.

Example 4.1. When we apply this process to the polytope $P = 5\Delta_4 - (1, 1, 1, 1)$ from Example 2.1, we get $X_P = \mathbb{P}^4$, and the hypersurface is our friend the quintic threefold $V \subseteq \mathbb{P}^4$.

If instead we use $P^\circ = \text{Conv}(e_1, e_2, e_3, e_4, (-1, -1, -1, -1))$, then we get the toric variety $X_{P^\circ} = \mathbb{P}^4/G$, where G is the group defined in Example 3.2. The six lattice points of P° consist of the origin and the five vertices, so that Equation (4) reduces to

$$c_0 + c_1 t_1 + c_2 t_2 + c_3 t_3 + c_4 t_4 + c_5 t_1^{-1} t_2^{-1} t_3^{-1} t_4^{-1} = 0.$$

When we apply the *homogenization* process described in ([12] Section 13) and ([4] §5.4), this equation becomes

$$c_0 x_0^5 + c_1 x_1^5 + c_2 x_2^5 + c_3 x_3^5 + c_4 x_4^5 + c_5 x_0 x_1 x_2 x_3 x_4 = 0.$$

Rescaling the coordinates x_0, \dots, x_5 appropriately, one can assume that $c_0 = \dots = c_4 = 1$. Hence we recover the Equation (3), which here defines the hypersurface of $X_{P^\circ} = \mathbb{P}^4/G$ that was denoted W/G in Example 3.2. As explained in that example, this gives the quintic mirror V° after a suitable resolution of singularities.

It is satisfying to see how P and P° lead naturally to the quintic threefold and its mirror. In 1994, Batyrev [2] (see also ([10] Section 4.1)) proved the following general results:

- A 4-dimensional reflexive polytope P gives a hypersurface in X_P that becomes a Calabi–Yau threefold V after a suitable resolution of singularities.
- The dual polytope P° gives a Calabi–Yau threefold V° , and the Hodge numbers of V and V° are related by

$$h^{11}(V^\circ) = h^{21}(V) \text{ and } h^{21}(V^\circ) = h^{11}(V),$$

as required by mirror symmetry.

In 1995, the paper [13] used reflexive polytopes to supply the “missing mirrors” mentioned at the beginning of this section. This was first clear indication of the deep relation between mirror symmetry and 4-dimensional reflexive polytopes. However, there are two things to keep in mind:

- Not all Calabi–Yau threefolds arise from Batyrev’s construction. We will learn more about this later in the paper.
- It is still an open question in physics as to whether V and V° give isomorphic SCFTs when P is an arbitrary 4-dimensional reflexive polytope. The evidence is compelling, but an actual isomorphism is only known for certain special cases, such as the quintic threefold and its mirror.

5. Duality and Symmetry in Mirror Symmetry

Given the themes of duality and symmetry, it makes sense to say more about what they mean in the context of mirror symmetry. The duality aspect is clear, given how P and P° give a mirror pair V and V° . But what about symmetry?

When considering a smooth projective threefold V , its Hodge numbers $h^{pq} = h^{pq}(V)$ from Equation (1) are often represented in the *Hodge diamond* shown in Figure 4.

is important in mathematical physics, where it shows up in various guises (central charge of a Virasoro algebra, $\pm 2 \times$ number of fermion generations, *etc.*). Mirror symmetry interchanges h^{11} and h^{21} . This leaves the vertical coordinate $h^{11} + h^{21}$ of Figure 5 unchanged but replaces the horizontal coordinate $\chi = 2(h^{11} - h^{21})$ with its negative. Hence mirror symmetry underlies the bilateral symmetry of Figure 5.

The 2015 paper [15] presents a state-of-the-art discussion of Figure 5, which includes data for other mirror pairs beyond those arising from 4-dimensional reflexive polytopes. That paper also describes the website [16] where the reader can find the most current version of Figure 5.

6. CICYs and Duality of Cones

We mentioned earlier that not all Calabi–Yau threefolds come from 4-dimensional reflexive polytopes. Currently, there is no classification of Calabi–Yau threefolds, although many constructions are known. Some involve toric varieties, while others, such as those described in ([10] Section 4.4), have nothing to do with toric methods.

Within the toric realm, there is more to the story than just hypersurfaces in 4-dimensional toric varieties. For example:

- A generic intersection of two cubic hypersurfaces in \mathbb{P}^5 is a Calabi–Yau threefold.
- A generic intersection of two quadric hypersurfaces and a cubic hypersurface in \mathbb{P}^6 is a Calabi–Yau threefold.

There are similar examples in higher dimensional projective spaces. These are example of *complete intersection Calabi–Yau threefolds*, often written CICYs. The term “complete intersection” refers to the fact that the number of defining equations equals the codimension.

These examples can be generalized to the toric setting where higher dimensional reflexive polytopes take center stage. The construction is based on work of Batyrev and Borisov. We will follow the version presented in [17], focusing on the case of CICY threefolds. The reader should consult [17] for references to the original papers.

Before we can begin, we need some tools from polyhedral geometry:

- The *Minkowski sum* of polytopes $P_1, P_2 \subseteq \mathbb{R}^d$ is defined by

$$P_1 + P_2 = \{u + v \mid u \in P_1, v \in P_2\}.$$

Note that $P_1 + P_2$ is a lattice polytope whenever P_1 and P_2 are.

- Points $m_1, \dots, m_s \in \mathbb{Z}^d$ generate the *rational convex polyhedral cone*

$$\sigma = \text{Cone}(m_1, \dots, m_s) = \{\lambda_1 m_1 + \dots + \lambda_s m_s \mid \lambda_i \geq 0\} \subseteq \mathbb{R}^d.$$

- Given such a cone σ , its *dual* is

$$\sigma^\vee = \{u \in \mathbb{R}^d \mid u \cdot m \geq 0 \text{ for all } m \in \sigma\}.$$

Then σ^\vee is again a rational convex polyhedral cone, and we have the duality $(\sigma^\vee)^\vee = \sigma$.

Proofs of these standard facts can be found in [3].

Now suppose that we have a $(r + 3)$ -dimensional reflexive polytope P that is a Minkowski sum

$$P = P_1 + \dots + P_r, \tag{5}$$

where each P_i is a lattice polytope containing the origin. This is called a *nef-partition* for reasons having to do with numerically effective divisors on toric varieties.

Given a nef-partition Equation (5), we get the toric variety X_P of dimension $r + 3$, and using the lattice points of the Minkowski summands P_1, \dots, P_r as in Equation (4), we get hypersurfaces $Y_1, \dots, Y_r \subseteq X_P$. If the equations of the Y_i are sufficiently generic, then the intersection

$$Y_1 \cap \dots \cap Y_r$$

is complete intersection threefold whose resolution of singularities (carefully done) is a Calabi–Yau threefold V . This is CICY threefold determined by the nef-partition Equation (5). The examples presented at the beginning of this section are instances of this construction.

To create the mirror family, we use duality, but we have to be careful since in Equation (5), the P_i contain 0, but not as an interior point. This means that the dual P_i° is an unbounded polyhedron. The key idea of the Batyrev–Borisov construction is to define “dual” polytopes Q_1, \dots, Q_r as follows:

$$Q_i = \{u \in \mathbb{R}^d \mid u \cdot m \geq -1 \text{ for all } m \in P_i, u \cdot m \geq 0 \text{ for all } m \in P_j, j \neq i\}. \tag{6}$$

Here are the key properties of Q_1, \dots, Q_r :

- Q_1, \dots, Q_r are lattice polytopes containing the origin.
- $Q = Q_1 + \dots + Q_r$ is a reflexive polytope of dimension $r + 3$.

In other words, $Q = Q_1 + \dots + Q_r$ is a nef-partition, called the *dual nef-partition*. This also works in reverse, since $P = P_1 + \dots + P_r$ is the dual of $Q = Q_1 + \dots + Q_r$.

The dual nef-partition $Q = Q_1 + \dots + Q_r$ gives the toric variety X_Q and the hypersurfaces coming from Q_1, \dots, Q_r . The resulting CICY threefold V° is a candidate for the mirror of the CICY V of the original nef-partition $P = P_1 + \dots + P_r$.

To get a better sense of what Equation (6) means from the point of view of duality, let us focus on Q_1 . First observe that the cone

$$\text{Cone}(P_2 \cup \dots \cup P_r) = \text{Cone}(P_2 + \dots + P_r)$$

is rational polyhedral since it is the cone generated by the vertices of P_2, \dots, P_r . Then we can write Q_1 as follows:

$$\begin{aligned} Q_1 &= \{u \in \mathbb{R}^d \mid u \cdot m \geq -1 \text{ for all } m \in P_1, u \cdot m \geq 0 \text{ for all } m \in P_j, j = 2, \dots, r\} \\ &= P_1^\circ \cap \text{Cone}(P_2 \cup \dots \cup P_r)^\vee. \end{aligned}$$

Individually, P_1° and $\text{Cone}(P_2 \cup \dots \cup P_r)^\vee$ are unbounded polyhedra. Their intersection is bounded, *i.e.*, is a polytope, because 0 is an interior point of $P_1 + \dots + P_r$. This works not just for Q_1 but for all

of the Q_i . The surprise, as noted in the above bullets, is that the Q_i are lattice polytopes with reflexive Minkowski sum.

The resulting “duality” between P and Q is remarkable: we take P and decompose it into pieces via $P = P_1 + \dots + P_r$. For each piece P_i , we modify the usual dual P_i° using the cone dual to the remaining pieces. This gives Q_i , and then we assemble the Q_i to create $Q = Q_1 + \dots + Q_r$.

Here is an example of what this looks like in an especially simple case.

Example 6.1. Consider the reflexive polygon $P = 3\Delta_2 - (1, 1) \subseteq \mathbb{R}^2$. This is the polygon labeled “9” in Figure 3. Note that P is the 2-dimensional analog of the polytope $5\Delta_4 - (1, 1, 1, 1) \subseteq \mathbb{R}^4$ that gives the quintic threefold.

Figure 6 shows a nef-partition of P and the resulting dual nef-partition. In the figure, note that Q_1 is a line segment. The “dual” Q is equivalent to the polygon labeled “5a” in Figure 3, while the usual dual P° of P is the polygon labeled “3” in Figure 3.

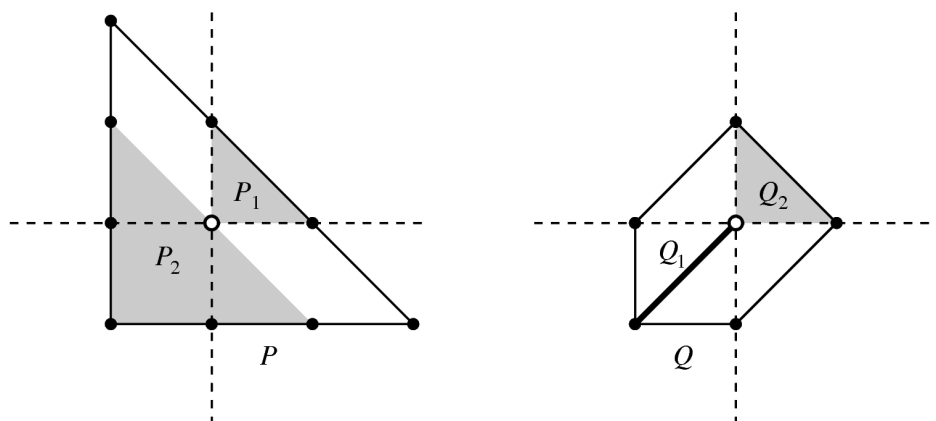


Figure 6. Dual nef-partitions $P = P_1 + P_2$ and $Q = Q_1 + Q_2$.

This example shows that the duality of nef-partitions differs from the usual duality of polytopes.

It is also possible to encode the duality of nef-partitions into the standard duality of cones. Given the nef-partition $P = P_1 + \dots + P_r$ in \mathbb{R}^d , consider the cone in $\mathbb{R}^d \times \mathbb{R}^r$ defined by

$$C_{\{P_i\}} = \text{Cone}(P_1 \times e_1) + \dots + \text{Cone}(P_r \times e_r) \subseteq \mathbb{R}^d \times \mathbb{R}^r,$$

and define $C_{\{Q_i\}}$ similarly. Then one can show that $C_{\{P_i\}}$ and $C_{\{Q_i\}}$ are dual cones under the standard dot product in $\mathbb{R}^d \times \mathbb{R}^r$.

The cones $C_{\{P_i\}}$ and $C_{\{Q_i\}}$ are examples of *dual reflexive Gorenstein cones*. The duality of these cones leads to additional examples relevant to mirror symmetry. We refer the reader to [17] for details and further reading.

7. Conclusions

In this article we have explored several types of duality of polytopes that relate to mirror symmetry. Although both physics and algebraic geometry have moved beyond the “classical” picture of mirror

symmetry presented here, we hope to have made a convincing demonstration that something wonderful happened in 1994 when Batyrev suggested that the duality of reflexive polytopes was relevant to mirror symmetry.

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Conflicts of Interest

The author declares no conflict of interest.

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