Strata of rational space curves

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Abstract

The $\boldsymbol{\mu}$-invariant $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$ of a rational space curve gives important information about the curve. In this paper, we describe the structure of all parameterizations that have the same $\boldsymbol{\mu}$-type, what we call a *$\boldsymbol{\mu}$-stratum*, and as well the closure of strata. Many of our results are based on papers by the second author that appeared in the commutative algebra literature. We also present new results, including an explicit formula for the codimension of the locus of non-proper parametrizations within each $\boldsymbol{\mu}$-stratum and a decomposition of the smallest $\boldsymbol{\mu}$-stratum based on which two-dimensional rational normal scroll the curve lies on.

1 Introduction

A rational curve of degree n in projective 3-space is parametrized by

$$F(s,t) = (a_0(s,t), a_1(s,t), a_2(s,t), a_3(s,t)) \tag{1.1}$$

where a_0, a_1, a_2, a_3 are relatively prime homogeneous polynomials of degree n. If F is generically one-to-one and a_0, a_1, a_2, a_3 are linearly independent, then the image curve C has degee n and does not lie in a plane, i.e., C is a genuine *space curve*.

For a parametrized planar curve of degree n, the 1998 paper [CSC] introduced the idea of a *μ-basis*. Since then, μ-bases have proved useful in the

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study of the singularities of rational plane curves, as evidenced by the papers [FWL, SCG] in the geometric modeling literature and [CKPU] in the commutative algebra literature.

The groundwork for the space curve case appears in [CSC, Sec. 5], and the connection with singularities has been studied in several papers, including [JWG, SC, SJG, WJG]. For references to the (fairly extensive) algebraic geometry literature on rational space curves, we direct the reader to the bibliography of [I3].

An idea introduced in [CSC] was to study the μ-stratum consisting of *all* parametrizations of plane rational curves with given degree and μ-type. In this paper, we will extend this idea to rational space curves and more generally to define the $\boldsymbol{\mu}$-strata of rational curves in projective d-space, based on the papers [I1, I2] of the second author. A terse version of the results presented in Sections 3 and 4.2, written for commutative algebraists, can be found in [I3]. The results concerning properness in Section 4.1 are new to this paper and this topic is not mentioned in [I1, I2]. Results concerning the decomposition of the smallest $\boldsymbol{\mu}$-stratum are geometric consequences of [I2], but were not developed there.

We will review the planar case in Section 2 and then discuss $\boldsymbol{\mu}$-strata in higher dimensions in Section 3. We will give examples to illustrate the unexpected behaviors that can arise. Section 4 will study non-proper parametrizations and explain how parametrizations in the smallest $\boldsymbol{\mu}$-stratum relate to two-dimensional rational normal scrolls in $\mathbb{P}^d$. Proofs will be given in Appendix A.

2 Planar Rational Curves

For the rest of the paper, we will work over an arbitrary infinite field k, which in practice is usually $\mathsf{k} = \mathbb{R}$ or $\mathbb{C}$. Set $R = \mathsf{k}[s,t]$ and let R_n be the subspace consisting of homogeneous polynomials of degree n.

A rational curve in the projective plane is parametrized by

$$F(s,t) = (a_0(s,t), a_1(s,t), a_2(s,t)), \tag{2.1}$$

where $a_0, a_1, a_2 \in R_n$. In this section, we will assume that a_0, a_1, a_2 are relatively prime and linearly independent and that F is generically one-to-one.

A *moving line* of degree m is a polynomial $A_0(s,t)x + A_1(s,t)y + A_2(s,t)z$ with $A_0, A_1, A_2 \in R_m$. It *follows* the parametrization if

$$A_0a_0 + A_1a_1 + A_2a_2 = 0 \text{ in } R. \tag{2.2}$$

A μ-*basis* for F consists of a pair of moving lines p, q that follow the parametrization and have the property that *any* moving line that follows the parametrization is a linear combination (with polynomial coefficients) of p and q. Assuming $\deg(p) \le \deg(q)$, one sets $\mu = \deg(p)$, so that $\deg(q) = n - \mu$ since it is known that $\deg(p) + \deg(q) = n$. In this situation, we say that F has *type* μ (this is the terminology used in [SJG]). Thus the μ-type is the minimum degree of a moving line that follows F. Note that

$$1 \le \mu \le \lfloor n/2 \rfloor$$

since a_0, a_1, a_2 are linearly independent and $\mu \le n - \mu$.

To connect this with algebraic geometry and commutative algebra, we introduce the ideal $I = \langle a_0, a_1, a_2 \rangle \subseteq R$. Then, as explained in [CSC], the Hilbert Syzygy Theorem gives an exact sequence

$$0 \longrightarrow R(-n-\mu) \oplus R(-2n+\mu) \xrightarrow{A} R(-n)^3 \xrightarrow{B} I \longrightarrow 0, \tag{2.3}$$

where B is given by (a_0, a_1, a_2) and A is the 3×2 matrix whose columns are the coefficients of p and q, and $BA = 0$. The notation $R(-n-\mu), R(-2n+\mu), R(-n)$ reflects the degree shifts needed to make the maps in (2.3) preserve degrees. We note that μ-bases and μ-types appeared in the algebraic geometry literature as early as 1986 (see [Asc1, Asc2]).

When we think of p and q as columns of the matrix A, then the Hilbert-Burch Theorem asserts that the cross product $p \times q$ is the parametrization (a_0, a_1, a_2), up to multiplication by a nonzero constant. This feature makes it easy to create parametrizations with given μ-type: just choose generic p and q of respective degrees μ and $n - \mu$ and take their cross product.

To study all parametrizations with the same μ-type, we begin with the subset $\mathcal{P}_n \subseteq R_n^3$ consisting of all relatively prime linearly independent triples (a_0, a_1, a_2) for which the parametrization is generically one-to-one. Then, for $1 \le \mu \le \lfloor n/2 \rfloor$, we have the *$\mu$-stratum*

$$\mathcal{P}_n^\mu = \{(a_0, a_1, a_2) \in \mathcal{P}_n \mid (a_0, a_1, a_2) \text{ has type } \mu\}.$$

In [CSC], it was proved that $\mathcal{P}_n^\mu$ is irreducible of dimension

$$\dim(\mathcal{P}_n^\mu) = \begin{cases} 3(n+1), & \text{if } \mu = \lfloor n/2 \rfloor, \\ 2n + 2\mu + 4, & \text{if } \mu < \lfloor n/2 \rfloor. \end{cases} \tag{2.4}$$

The μ-stratum $\mathcal{P}_n^\mu$ is not closed in $\mathcal{P}_n$. Let $\overline{\mathcal{P}}_n^\mu$ denote its Zariski closure in $\mathcal{P}_n$. In [CSC], it was conjectured that

$$\overline{\mathcal{P}}_n^\mu = \mathcal{P}_n^1 \cup \cdots \cup \mathcal{P}_n^\mu. \tag{2.5}$$

This was proved in 2004 by D'Andrea [D]. This result also is a consequence of the 1977 memoir [I1] or the 2004 article [I2] by the second author, though these are written from a very different viewpoint. It was eight years after [D] appeared before a connection was made between these papers.

Here is the intuition behind (2.4) and (2.5):

- (2.4) says that the smaller the μ, the more special the parametrization.
- (2.5) says that if we are moving around in $\mathcal{P}_n^\mu$ and reach its boundary, then with high probability, we hit a point of $\mathcal{P}_n^{\mu-1}$, i.e., μ drops by one unless we are really unlucky.

We will see in the next section that the results in [I1, I2] also apply to parametrizations in projective d-space, though the analogs of the above two bullets become more sophisticated as d increases.

3 Rational Curves in Projective Space

Curves in the plane and 3-dimensional space are the most important to geometric modeling. Since the results of [I1, I2] apply to curves in projective d-space for all $d \geq 1$, we will work in this greater generality. One way to think of our approach is that it gives a unified treatment of rational plane and space curves, as well as rational curves in higher dimensions.

We therefore start with $d+1$ homogeneous polynomials $a_0, \ldots, a_d \in R_n$. Then the function

$$F(s,t) = (a_0(s,t), \ldots, a_d(s,t))$$

parametrizes a curve in projective d-space, generalizing (1.1) and (2.1). We will assume that $a_0, \ldots, a_d$ are linearly independent, which implies that the image curve is not contained in a hyperplane. Note also that the span $\mathrm{Span}(a_0, \ldots, a_d)$ is a $(d+1)$-dimensional subspace of R_n. This is important, since the results of [I1, I2] for $R = \mathsf{k}[s,t]$ are stated in terms of subspaces of R_n of a given dimension, equal to $d+1$ in our situation. We will say more about this in Appendix A.

In Section 2, we made two assumptions beyond linear independence:

- The parametrization is proper, i.e., generically one-to-one, and
- The polynomials in the parametrization are relatively prime.

In this section, we will dispense with the first assumption, so that we allow non-proper parametrizations. We will see in Section 4.1 that this is harmless. As for the second assumption, we will give two versions of our main results, one that assumes relatively prime, and one that does not.

3.1 $\boldsymbol{\mu}$-Bases and $\boldsymbol{\mu}$-Types

Before stating our results, we need to define $\boldsymbol{\mu}$-types and $\boldsymbol{\mu}$-bases.

Proposition 3.1. *Let $a_0, \ldots, a_d \in R_n$ be linearly independent and let*

$$I = \langle a_0, \ldots, a_d \rangle \subseteq R$$

be the the ideal generated by $a_0, \ldots, a_d$. Then there exist integers $\mu_1, \ldots, \mu_d \geq 1$ and an exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^d R(-n-\mu_i) \xrightarrow{A} R(-n)^{d+1} \xrightarrow{B} I \longrightarrow 0. \tag{3.1}$$

Furthermore, if we set $h = \gcd(a_0, \ldots, a_d)$, then $\mu_1 + \cdots + \mu_d = n - c$, $c = \deg(h)$, and $B = (a_0, \ldots, a_d)$ consists of h times the maximal minors of A (up to sign).

Proof. Let $h = \gcd(a_0, \ldots, a_d)$ and set $b_i = a_i/h$. Then $\gcd(b_0, \ldots, b_d) = 1$ and $\deg(b_i) = n-c$ since $c = \deg(h)$. The results of [CSC, Sec. 5] apply to $b_0, \ldots, b_d$, so that we have an exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^d R(-n-c-\mu_i) \xrightarrow{A} R(-n-c)^{d+1} \xrightarrow{\widetilde{B}} \langle b_0, \ldots, b_d \rangle \longrightarrow 0,$$

where $\widetilde{B} = (b_0, \ldots, b_d)$ and A is a $(d+1) \times d$ matrix whose columns form a basis of the module of moving hyperplanes $A_0x_0 + \cdots + A_dx_d$ that follow the parametrization given by $(b_0, \ldots, b_d)$. Since $\deg(b_i) = n - c$, [CSC] also implies that $\mu_1 + \cdots + \mu_d = n - c$. Since $a_i = hb_i$ and $h \neq 0$, we have

$$\sum_{i=1}^{d} A_i a_i = 0 \iff \sum_{i=1}^{d} A_i b_i = 0.$$

Setting $B = h\widetilde{B}$ gives the exact sequence (3.1). The b_i are (up to sign) the maximal minors of A by [CSC], and the final assertion of the proposition follows. □

If we require $\mu_1 \leq \cdots \leq \mu_d$, then the d-tuple

$$\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d) \tag{3.2}$$

is uniquely determined by $(a_0, \ldots, a_d)$. We call (3.2) the $\boldsymbol{\mu}$*-type* of $(a_0, \ldots, a_d)$, and the columns of the matrix A in (3.1) form a $\boldsymbol{\mu}$*-basis*. Note that when $d = 2$ and $\gcd(a_0, a_1, a_2) = 1$, Proposition 3.1 tells us that the $\boldsymbol{\mu}$-type can be written

$$\boldsymbol{\mu} = (\mu, n - \mu).$$

Hence we recover the μ-type in the planar case when the polynomials are relatively prime.

Remark 3.2. We use "$\boldsymbol{\mu}$" in two ways in this paper. When followed by a hyphen, as in $\boldsymbol{\mu}$-type, $\boldsymbol{\mu}$-basis or $\boldsymbol{\mu}$-stratum, the $\boldsymbol{\mu}$ is part of the notation and has no specific value. But when used by itself, $\boldsymbol{\mu}$ denotes a vector of integers, such as $\boldsymbol{\mu} = (1, 2, 3)$.

3.2 The Relatively Prime Case

To state our first result, let

$$\mathcal{P}_{n,d} \subseteq R_n^{d+1}$$

consist of all $(a_0, \ldots, a_d) \in R_n^{d+1}$ such that $a_0, \ldots, a_d$ are linearly independent and relatively prime. Linear independence means that the image of the parametrization does not lie in any hyperplane of $\mathbb{P}^d$. We will always assume that $n \geq d$ (otherwise $\mathcal{P}_{n,d}$ is empty) and $d \geq 2$ (curves in $\mathbb{P}^1$ are not interesting). One can show that $\mathcal{P}_{n,d}$ is a nonempty Zariski open subset of R_n^{d+1}, i.e., the complement of a proper closed subvariety of R_n^{d+1}.

Given integers with $1 \leq \mu_1 \leq \cdots \leq \mu_d$, we set $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d)$ and $|\boldsymbol{\mu}| = \mu_1 + \cdots + \mu_d$. We call $\boldsymbol{\mu}$ a *d-part partition.* (Many references write partitions in descending order, e.g., $7 = 4 + 2 + 1$. We use ascending order since this is how $\boldsymbol{\mu}$-bases are written in the geometric modeling literature.)

If $\boldsymbol{\mu}$ is a d-part partition of n (so $n = |\boldsymbol{\mu}|$), then we define the $\boldsymbol{\mu}$*-stratum*

$$\mathcal{P}_{n,d}^{\boldsymbol{\mu}} = \{(a_0, \ldots, a_d) \in \mathcal{P}_{n,d} \mid (a_0, \ldots, a_d) \text{ has type } \boldsymbol{\mu}\}. \tag{3.3}$$

This set has the following structure.

Theorem 3.3. *Assume that $\boldsymbol{\mu}$ is a d-part partition of n. Then $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ is a Zariski open subset of a subvariety of $R_n^{d+1} \simeq \mathsf{k}^{(d+1)(n+1)}$. Furthermore, $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ is irreducible of dimension*

$$\dim(\mathcal{P}^{\boldsymbol{\mu}}_{n,d}) = (d+1)(n+1) - \sum_{i>j} \max(0, \mu_i - \mu_j - 1).$$

The proofs of all theorems in this section will be given in Appendix A.

Example 3.4. In the plane case, we have $d = 2$ and $\boldsymbol{\mu} = (\mu, n-\mu)$. Here, $\mathcal{P}^{\boldsymbol{\mu}}_{n,d} = \mathcal{P}^{\mu}_{n}$ as defined in Section 2, and then Theorem 3.3 implies that

$$\begin{aligned}\dim(\mathcal{P}^{\mu}_{n}) &= 3(n+1) - \max(0, (n-\mu) - \mu - 1)\\ &= \begin{cases} 3(n+1), & \text{if } n-\mu = \mu, \text{ i.e., } \mu = \lfloor n/2 \rfloor,\\ 2n+2\mu+4, & \text{if } n-\mu > \mu, \text{ i.e., } \mu < \lfloor n/2 \rfloor,\end{cases}\end{aligned}$$

in agreement with (2.4).

Example 3.5. A space curve case studied in [JWG] is $\boldsymbol{\mu} = (1,1,n-2)$. Assuming $n \geq 4$, one computes that

$$\dim(\mathcal{P}^{\boldsymbol{\mu}}_{n,3}) = 4(n+1) - 2\max(0, (n-2) - 1 - 1) = 2n + 12.$$

We will soon see that this is the smallest $\boldsymbol{\mu}$-stratum of $\mathcal{P}_{n,3}$.

In general, $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ is not a subvariety of $\mathcal{P}_{n,d}$. We let $\overline{\mathcal{P}}^{\boldsymbol{\mu}}_{n,d}$ denote its Zariski closure in $\mathcal{P}_{n,d}$, i.e., the smallest subvariety of $\mathcal{P}_{n,d}$ containing $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$. Theorem 3.3 tells us that $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ is Zariski open in $\overline{\mathcal{P}}^{\boldsymbol{\mu}}_{n,d}$. The theorem also implies that $\overline{\mathcal{P}}^{\boldsymbol{\mu}}_{n,d}$ is irreducible with

$$\dim(\overline{\mathcal{P}}^{\boldsymbol{\mu}}_{n,d}) = \dim(\mathcal{P}^{\boldsymbol{\mu}}_{n,d}).$$

The expectation is that the complement $\overline{\mathcal{P}}^{\boldsymbol{\mu}}_{n,d} \backslash \mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ should consist of "smaller" $\boldsymbol{\mu}$-strata. We compare different $\boldsymbol{\mu}$-types as follows.

Definition 3.6. Given d-part partitions $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$, we define $\boldsymbol{\mu} \leq \boldsymbol{\mu}'$ provided

$$\mu_1 \leq \mu_1', \ \mu_1 + \mu_2 \leq \mu_1' + \mu_2', \ \ldots, \ \mu_1 + \cdots + \mu_d \leq \mu_1' + \cdots + \mu_d'.$$

Note that $\boldsymbol{\mu} \leq \boldsymbol{\mu}'$ implies $|\boldsymbol{\mu}| \leq |\boldsymbol{\mu}'|$.

We can now describe the Zariski closure of $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ in $\mathcal{P}_{n,d}$. Recall from the discussion leading up to (3.3) that all $\boldsymbol{\mu}$-strata occuring in $\mathcal{P}_{n,d}$ satisfy $|\boldsymbol{\mu}| = n$.

Theorem 3.7. *The Zariski closure of $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ in $\mathcal{P}_{n,d}$ is $\overline{\mathcal{P}}^{\boldsymbol{\mu}}_{n,d} = \bigcup_{\boldsymbol{\mu}' \leq \boldsymbol{\mu}} \mathcal{P}^{\boldsymbol{\mu}'}_{n,d}$.*

Since $\boldsymbol{\mu}$-strata are disjoint, this theorem implies that $\overline{\mathcal{P}}^{\boldsymbol{\mu}}_{n,d} \backslash \mathcal{P}^{\boldsymbol{\mu}}_{n,d} = \bigcup_{\boldsymbol{\mu}' < \boldsymbol{\mu}} \mathcal{P}^{\boldsymbol{\mu}'}_{n,d}$, confirming our intuition that $\overline{\mathcal{P}}^{\boldsymbol{\mu}}_{n,d} \setminus \mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ is a union of smaller strata.

Example 3.8. Since $(\mu, n-\mu) \leq (\mu', n-\mu')$ if and only if $\mu \leq \mu'$, we see that Theorem 3.7 reduces to (2.5) when $d = 2$.

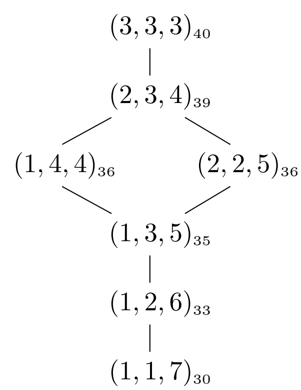
Example 3.9. One easily checks that $(1, 1, n-2) \leq \boldsymbol{\mu}$ for all 3-part partitions $\boldsymbol{\mu}$ of n. This and Theorem 3.7 justify our earlier claim that $\mathcal{P}_{n,3}^{(1,1,n-2)}$ is the smallest $\boldsymbol{\mu}$-stratum of $\mathcal{P}_{n,3}$. Proposition 3.12 will describe the smallest $\boldsymbol{\mu}$-stratum of $\mathcal{P}_{n,d}$.

Example 3.10. Sextic curves in dimension 3 have been studied in [JWG]. Here, the stratification is especially simple since the 3-part partitions of 6 are given by $(1, 1, 4) \leq (1, 2, 3) \leq (2, 2, 2)$. Hence

$$\mathcal{P}_{6,3} = \mathcal{P}_{6,3_{28}}^{(2,2,2)} \cup \mathcal{P}_{6,3_{27}}^{(1,2,3)} \cup \mathcal{P}_{6,3_{24}}^{(1,1,4)},$$

and the Zariski closure of each stratum consists of that stratum together with those to the right of it in the above union. The small subscript gives the dimension of each stratum.

Example 3.11. When $d = 3$, the smallest n for which incomparable $\boldsymbol{\mu}$-types exist is $n = 9$, and the types in question are $(1, 4, 4)$ and $(2, 2, 5)$. This gives the following stratification of $\mathcal{P}_{9,3}$:



Here, we have written $\mathcal{P}_{9,3_{\mathrm{dim}}}^{(\mu_1,\mu_2,\mu_3)}$ more simply as $(\mu_1, \mu_2, \mu_3)_{\mathrm{dim}}$, where "dim" gives the dimension of the stratum. The closure of a stratum consists of the stratum and everything strictly below it in the diagram.

One consequence of the diagram is that if we move around in $\mathcal{P}_{9,3}^{(2,3,4)}$ and reach its boundary, then with high probability we hit a point in *either* $\mathcal{P}_{9,3}^{(2,2,5)}$ or $\mathcal{P}_{9,3}^{(1,4,4)}$, and these possibilities are equally likely since both have codimension 3 in $\overline{\mathcal{P}}_{9,3}^{(2,3,4)}$.

3.3 The Largest and Smallest Strata

The stratification of $\mathcal{P}_{9,3}$ shown in Example 3.11 has $\boldsymbol{\mu} = (3, 3, 3)$ at the top: it is the maximum stratum in the partial order of Definition 3.6 and it is also the unique stratum having the largest dimension 40. The stratum $\boldsymbol{\mu} = (1,1,7)$ at the bottom is the minimum in the partial order of Definition 3.5 and also the unique stratum of smallest dimension 20. This generalizes as follows.

Proposition 3.12. *Given integers $n \geq d \geq 2$, write $n = kd + r$ where $k, r \in \mathbb{Z}$ and $0 \leq r < d$. Then any d-part partition $\boldsymbol{\mu}$ of n satisfies*

$$\boldsymbol{\mu}_{\min} = (\underbrace{1,\dots,1}_{d-1}, n-d+1) \leq \boldsymbol{\mu} \leq (\underbrace{k,\dots,k}_{d-r}, \underbrace{k+1,\dots,k+1}_{r}) = \boldsymbol{\mu}_{\max}.$$

Furthermore:

1. $\dim(\mathcal{P}_{n,d}^{\boldsymbol{\mu}_{\max}}) = (d+1)(n+1)$.
2. $\boldsymbol{\mu}_{\min} = \boldsymbol{\mu}_{\max}$ *if and only if* $n = d$ *or* $n = d+1$.
3. *If* $n \geq d+1$, *then* $\dim(\mathcal{P}_{n,d}^{\boldsymbol{\mu}_{\min}}) = d^2 + d + 2n$ *and* $\mathcal{P}_{n,d}^{\boldsymbol{\mu}_{\min}}$ *has codimension* $(d-1)(n-d-1)$ *in* $\mathcal{P}_{n,d} = \overline{\mathcal{P}}_{n,d}^{\boldsymbol{\mu}_{\max}}$.

Proof. The formulas for $\dim(\mathcal{P}_{n,d}^{\boldsymbol{\mu}_{\min}})$ and $\dim(\mathcal{P}_{n,d}^{\boldsymbol{\mu}_{\max}})$ follow from Theorem 3.3. We omit the rest of the straightforward proof. □

Applied to $\mathcal{P}_{9,3}$, this proposition gives $\boldsymbol{\mu}_{\max} = (3,3,3)$ and $\boldsymbol{\mu}_{\min} = (1,1,7)$. Furthermore, $\mathcal{P}_{9,3}^{(1,1,7)}$ has dimension $3^2 + 3 + 2\cdot 9 = 30$, and its codimension in $\mathcal{P}_{n,d} = \overline{\mathcal{P}}_{9,3}^{(3,3,3)}$ is $(3-1)(9-3-1) = 10$, in agreement with Example 3.11.

We will say more about the structure of $\mathcal{P}_{n,d}^{\boldsymbol{\mu}_{\min}}$ in Section 4.2.

3.4 The General Case

We can also allow $a_0, \dots, a_d$ to have a common factor. Let

$$\mathcal{CP}_{n,d} \subseteq R_n^{d+1}$$

consist of all $(a_0, \dots, a_d) \in R_n^{d+1}$ such that $a_0, \dots, a_d$ are linearly independent. This has $\mathcal{P}_{n,d}$ as an open subset and in addition contains those parametrizations where $(a_0, \dots, a_d)$ have a nontrivial common factor. Recall from Proposition 3.1 that the $\boldsymbol{\mu}$-type of $(a_0, \dots, a_d) \in \mathcal{CP}_{n,d}$ satisfies $|\boldsymbol{\mu}| = n - c$, where $\deg(\gcd(a_0, \dots, a_d)) = c$.

Given a d-part partition $\boldsymbol{\mu}$ with $|\boldsymbol{\mu}| \leq n$, we have the $\boldsymbol{\mu}$-stratum

$$\mathcal{P}_{n,d}^{\boldsymbol{\mu}} = \{(a_0, \dots, a_d) \in \mathcal{CP}_{n,d} \mid (a_0, \dots, a_d) \text{ has type } \boldsymbol{\mu}\}.$$

Since a d-part partition satisfies $|\boldsymbol{\mu}| \geq d$, we will always assume that $d \leq |\boldsymbol{\mu}| \leq n$. Note also that the $\boldsymbol{\mu}$-stratum $\mathcal{P}_{n,d}^{\boldsymbol{\mu}}$ lies in $\mathcal{P}_{n,d}$ (i.e., is one of the strata (3.3)) if and only if $|\boldsymbol{\mu}| = n$.

Theorem 3.13. *Let $\boldsymbol{\mu}$ be a d-part partition satisfying $d \leq |\boldsymbol{\mu}| \leq n$. Then $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ is a Zariski open subset of a subvariety of R_n^{d+1}. Furthermore, $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ is irreducible of dimension*

$$\dim(\mathcal{P}^{\boldsymbol{\mu}}_{n,d}) = d|\boldsymbol{\mu}| + d + n + 1 - \sum_{i>j} \max(0, \mu_i - \mu_j - 1).$$

We prove this in Appendix A using results from [I1, I2]. Here we give an intuitive argument to explain the formula for $\dim(\mathcal{P}^{\boldsymbol{\mu}}_{n,d})$. Multiplication gives a surjection

$$(R_{n-|\boldsymbol{\mu}|} \setminus \{0\}) \times \mathcal{P}^{\boldsymbol{\mu}}_{|\boldsymbol{\mu}|,d} \longrightarrow \mathcal{P}^{\boldsymbol{\mu}}_{n,d}$$

that maps relatively prime d-tuples of degree $|\boldsymbol{\mu}|$ to d-tuples of degree n with a common factor h of degree $n - |\boldsymbol{\mu}|$. The fibers of this map have dimension 1 since

$$(\lambda^{-1}h)(\lambda b_0, \ldots, \lambda b_d) = h(b_0, \ldots, b_n) \text{ for all } \lambda \in \mathsf{k} \setminus \{0\}$$

and gcd's are well-defined only up to multiplication by a nonzero constant. By Theorem 3.3, it follows that

$$\dim(\mathcal{P}^{\boldsymbol{\mu}}_{n,d}) = \big(n - |\boldsymbol{\mu}| + 1\big) + \big((d+1)(|\boldsymbol{\mu}| + 1) - \sum_{i>j} \max(0, u_i - u_j - 1)\big) - 1.$$

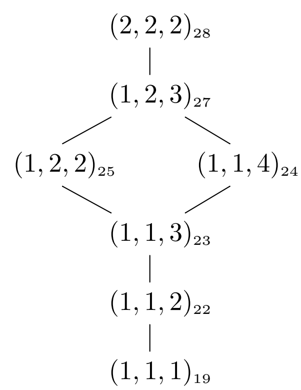
This easily reduces to the formula given in Theorem 3.13.

We can also compute the Zariski closure of $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ in $\mathcal{CP}_{n,d}$.

Theorem 3.14. *The Zariski closure of $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ in $\mathcal{CP}_{n,d}$ is $\overline{\mathcal{P}}^{\boldsymbol{\mu}}_{n,d} = \bigcup_{\boldsymbol{\mu}' \leq \boldsymbol{\mu}} \mathcal{P}^{\boldsymbol{\mu}'}_{n,d}$, where the union is over all d-part partitions $\boldsymbol{\mu}'$ with $d \leq |\boldsymbol{\mu}'| \leq n$ and $\boldsymbol{\mu}' \leq \boldsymbol{\mu}$.*

By definition, $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ consists of parametrizations with a common factor of degree $n - |\boldsymbol{\mu}|$. Theorem 3.14 tells us that in $\mathcal{CP}_{n,d}$, the difference $\overline{\mathcal{P}}^{\boldsymbol{\mu}}_{n,d} \setminus \mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ consists of strata $\overline{\mathcal{P}}^{\boldsymbol{\mu}'}_{n,d}$, where $\boldsymbol{\mu}' \leq \boldsymbol{\mu}$. Since $\boldsymbol{\mu}' \leq \boldsymbol{\mu}$ implies that $n - |\boldsymbol{\mu}'| \geq n - |\boldsymbol{\mu}|$, we see that $\overline{\mathcal{P}}^{\boldsymbol{\mu}}_{n,d}$ may include strata with common factors of larger degree.

Example 3.15. In Example 3.10, we saw that $\mathcal{P}_{6,3}$ has a simple stratification with three strata. These and four other strata appear in the more complex stratification of $\mathcal{CP}_{6,3}$:



Here, we write $\mathcal{P}^{(\mu_1,\mu_2,\mu_3)}_{6,3_{\mathrm{dim}}}$ as $(\mu_1,\mu_2,\mu_3)_{\mathrm{dim}}$, similar to Example 3.11.

This diagram tells us that if we move around in $\mathcal{P}^{(1,2,3)}_{6,3}$ and reach its boundary in $\mathcal{CP}_{6,3}$, then with high probability we hit a point in *either* $\mathcal{P}^{(1,2,2)}_{6,3}$ (acquire a common factor) or $\mathcal{P}^{(1,1,4)}_{6,3}$ (remain relatively prime). These possibilities are not equally likely since the former has codimension 2 in $\overline{\mathcal{P}}^{(1,2,3)}_{6,3}$ while the latter has codimension 3.

4 Further Topics

In this section we investigate non-proper parametrizations and look more closely at the smallest and largest strata of $\mathcal{P}_{n,d}$. Section 4.1 is new to this paper; Section 4.2 is a geometric interpretation of some results of [I2].

4.1 Non-Proper Parametrizations

In this section, we restrict our attention to $\boldsymbol{\mu}$-strata $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ with $|\boldsymbol{\mu}| = n$, i.e., $\boldsymbol{\mu}$ is a d-part partition of n. This means that all $(a_0,\ldots,a_d) \in \mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ are relatively prime. As in Section 3, we assume $n \geq d \geq 2$.

A parametrization $(a_0,\ldots,a_d) : \mathbb{P}^1 \to \mathbb{P}^d$ is proper if it is birational onto its image. In general, let k be number of points in the preimage of a generic point in the image. We say that the parametrization has *generic degree* k. It is well-known that if the $a_i \in R_n$ are relatively prime, then

$$n = km, \tag{4.1}$$

where k is the generic degree of the parametrization and m is the degree of the image curve $C \subseteq \mathbb{P}^d$. Thus a proper parametrization has generic degree 1 and parametrizes a curve of degree n.

Proposition 4.1. *Let $k > 1$ be an integer. Then $\mathcal{P}_{n,d}$ contains a parametrization of generic degree k if and only if $k \mid n$ and $n \geq kd$.*

Proof. The restriction $k \mid n$ is obvious from (4.1). To understand the inequality $n \geq kd$, recall our assumption that the a_i are linearly independent, i.e., the image curve C does not lie in any hyperplane of $\mathbb{P}^d$. But C has degree $m = n/k$, and if $m < d$, then the $d+1$ polynomials of degree m parametrizing C would have to be linearly dependent, forcing C to lie in a hyperplane. Thus $n/k = m \geq d$. This proves one direction of the proposition; the proof of the other direction will be deferred until Section A.3. □

We next relate non-proper parametrizations to the $\boldsymbol{\mu}$-stratification of $\mathcal{P}_{n,d}$.

Proposition 4.2. *Suppose we have a $\boldsymbol{\mu}$-stratum $\mathcal{P}_{n,d}^{\boldsymbol{\mu}}$ with $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d)$ and let $k > 1$ be an integer. Then $\mathcal{P}_{n,d}^{\boldsymbol{\mu}}$ contains parametrizations of generic degree k if and only if $k \mid \mu_i$ for all i.*

The proof will be given in Section A.3. Proposition 4.2 has the following useful corollary.

Corollary 4.3. *Given $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d)$, the $\boldsymbol{\mu}$-stratum $\mathcal{P}_{n,d}^{\boldsymbol{\mu}}$ consists entirely of proper parametrizations if and only if $\gcd(\mu_1, \ldots, \mu_d) = 1$.*

Example 4.4. Suppose that $n = 12$ and $d = 3$. The integers $k > 1$ dividing 12 are $k = 2, 3, 4, 6, 12$. By Proposition 4.1, $\mathcal{P}_{12,3}$ has no parametrizations of generic degree $k = 6$ or 12 since $d = 3$, while $k = 2$, 3 and 4 can occur.

One can compute that $\mathcal{P}_{12,3}$ decomposes into 12 $\boldsymbol{\mu}$-strata, eight of which satisfy the gcd criterion of Corollary 4.3 and hence have no non-proper parametrizations. For the remaining four $\boldsymbol{\mu}$-strata, we have non-proper parametrizations of the following types:

- Generic degree 4 occurs in $\mathcal{P}_{12,3}^{(4,4,4)}$.
- Generic degree 3 occurs in $\mathcal{P}_{12,3}^{(3,3,6)}$.
- Generic degree 2 occurs in $\mathcal{P}_{12,3}^{(4,4,4)}$, $\mathcal{P}_{12,3}^{(2,4,6)}$, and $\mathcal{P}_{12,3}^{(2,2,8)}$.

One expects non-proper parametrizations to be rare. Our next task is to quantify this intuition by computing the size of the generic degree k locus in each $\boldsymbol{\mu}$-stratum. When $\mathcal{P}_{n,d}^{\boldsymbol{\mu}}$ contains a parametrization of generic degree $k > 1$, Proposition 4.2 implies that its $\boldsymbol{\mu}$-type can be written as $\boldsymbol{\mu} = k(\tilde{\mu}_1, \ldots, \tilde{\mu}_d)$, i.e., $\boldsymbol{\mu}$ is divisible by k. This implies $k \mid n$ since $n = k\tilde{\mu}_1 + \cdots + k\tilde{\mu}_d$.

Theorem 4.5. *Assume that $\boldsymbol{\mu}$ is divisible by $k > 1$. Then the parametrizations in $\mathcal{P}_{n,d}^{\boldsymbol{\mu}}$ of generic degree k form a nonempty constructible subset of $\mathcal{P}_{n,d}^{\boldsymbol{\mu}}$ with irreducible Zariski closure of codimension*

$$(k-1)(m(d+1) - S - 2), \quad S = \sum_{i>j} \max(0, \tilde{\mu}_i - \tilde{\mu}_j), \tag{4.2}$$

where $m = n/k$ and $\boldsymbol{\mu} = k(\tilde{\mu}_1, \ldots, \tilde{\mu}_d)$. Furthermore:

1. *The codimension is at least $(k-1)(d(d-1)+2m-2)$ and is always positive.*

2. A generic parametrization in $\mathcal{P}_{n,d}^{\boldsymbol{\mu}}$ *is proper.*

The proof of this theorem will be given in Section A.3. Here is a sketch of some of the ideas involved. Using Lüroth's Theorem, we will show that a parametrization $F : \mathbb{P}^1 \to \mathbb{P}^d$ of generic degree k is a composition

$$\mathbb{P}^1 \xrightarrow{G} \mathbb{P}^1 \xrightarrow{H} \mathbb{P}^d,$$

where G is defined by $(\alpha(s,t), \beta(s,t)) \in R_k \times R_k$ of degree k and H is a parametrization in $\mathcal{P}_{m,d}^{\tilde{\boldsymbol{\mu}}}$ of generic degree 1 for $\tilde{\boldsymbol{\mu}} = (\tilde{\mu}_1, \ldots, \tilde{\mu}_d)$. It will follow that composition gives a map

$$\varphi : \mathcal{P}_{m,d}^{\tilde{\boldsymbol{\mu}}} \times R_k \times R_k \longrightarrow \mathcal{P}_{n,d}^{\boldsymbol{\mu}}$$

(we have to shrink the domain slightly to make this work) whose image consists of parametrizations of generic degree k. The nonempty fibers of this map have dimension 4 coming from a natural action of $\mathrm{GL}(2, \mathsf{k})$. Hence the generic degree k locus has dimension

$$\dim(\mathcal{P}_{m,d}^{\tilde{\boldsymbol{\mu}}}) + 2(k+1) - 4 = (d+1)(m+1) - \sum_{i>j} \max(0, \tilde{\mu}_i - \tilde{\mu}_j - 1) + 2(k-1).$$

where we have used Theorem 3.3. The codimension formula in Theorem 4.5 follows by combining this with the corresponding formula for $\dim(\mathcal{P}_{n,d}^{\boldsymbol{\mu}})$. Full details will be provided in Section A.3.

Example 4.6. In Example 3.11, we saw that the maximum $\boldsymbol{\mu}$-stratum of $\mathcal{P}_{9,3}$ in the order of Definition 3.6 is $\mathcal{P}_{9,3}^{(3,3,3)}$ of dimension 40 and the minimum is $\mathcal{P}_{9,3}^{(1,1,7)}$ of dimension 20. The only non-proper parametrizations have generic degree 3 and lie in $\mathcal{P}_{9,3}^{(3,3,3)}$. Since $\tilde{\boldsymbol{\mu}} = (1,1,1)$ and $m = n/k = 3$ in this case, Theorem 4.5 implies that the generic degree 3 locus has codimension

$$(k-1)(m(d+1) - S - 2) = (3-1)(3 \cdot (3+1) - 0 - 2) = 2 \cdot 10 = 20$$

in $\mathcal{P}_{9,3}^{(3,3,3)}$. Hence non-proper parametrizations really are rare! Note also that

$$20 = (3-1)(3(3-1) + 2 \cdot 3 - 2) = (k-1)(d(d-1) + 2m - 2)$$

since $k = d = m = 3$. This shows that the lower bound in Theorem 4.5(1) is sharp.

Example 4.7. In Example 4.4 we noted that non-proper parametrizations with generic degrees $k = 4, 3$ and 2 occur in $\mathcal{P}_{12,3}$. We give the respective codimensions using the formula (4.2) of Theorem 4.5:

- Generic degree 4 occurs in $\mathcal{P}_{12,3}^{(4,4,4)}$, which has dimension 52. Here, we have $m = 3$ and $S = 0$, so that the non-proper codimension is $(4-1)(3 \cdot 4 - 2) = 30$. In other words, the non-proper parametrizations have dimension 22.

- Generic degree 3 occurs in $\mathcal{P}_{12,3}^{(3,3,6)}$, which has dimension 48. Here, $m = 4$ and $S = 2$ so the non-proper codimension is $(3-1)(4 \cdot 4 - 2 - 2) = 24$. Hence the non-proper locus has dimension 24.
- Generic degree 2 occurs in $\mathcal{P}_{12,3}^{(4,4,4)}$, $\mathcal{P}_{12,3}^{(2,4,6)}$, and $\mathcal{P}_{12,3}^{(2,2,8)}$. A similar calculation gives respective non-proper codimensions 22, 18, and 18 in $\boldsymbol{\mu}$-strata of dimensions 52, 47, and 42. So these non-proper strata have dimensions 30, 29 and 24, respectively. Again, the non-proper loci have very high codimensions.

4.2 The Smallest Stratum and Rational Normal Scrolls

In this section we describe a further stratification of the smallest $\boldsymbol{\mu}$-stratum $\boldsymbol{\mu}_{\min} = (1, \ldots, 1, n-d+1)$ in the relatively prime case, assuming $n \geq d+1$. The stratification will involve finding which rational normal scroll the curve lies on. We begin with an example.

Example 4.8. Rational curves in $\mathbb{P}^3$ with $\boldsymbol{\mu} = (1, 1, n-2)$ were studied in [WJG]. Corollary 6.8 of that paper uses $\boldsymbol{\mu}$-bases to show that when $n \geq 4$, such curves are either smooth or have a unique singular point of multiplicity $n-2$.

This result can be explained using Section 2 of the paper [KPU], which considers $\boldsymbol{\mu}_{\min} = (1, \ldots, 1, n-d+1)$ from a commutative algebra point of view. When $\boldsymbol{\mu} = (1, 1, n-2)$, the results of [KPU] imply that after a suitable change of coordinates in $\mathbb{P}^3$, the 4×3 matrix A from (3.1) can be assumed to be either

$$A = \begin{pmatrix} s & 0 & r_0 \\ t & 0 & r_1 \\ 0 & s & r_2 \\ 0 & t & r_3 \end{pmatrix}, \quad \deg(r_i) = n-2, \tag{4.3}$$

or

$$A = \begin{pmatrix} s & 0 & r_0 \\ t & s & r_1 \\ 0 & t & r_2 \\ 0 & 0 & r_3 \end{pmatrix}, \quad \deg(r_i) = n-2. \tag{4.4}$$

(See [KPU, Prop. 2.1].) In either case, the first column of A gives the moving plane $sx_0 + tx_1 = 0$ which contains the line $L_1 = \{(0,0,a,b) \mid (a,b) \in \mathbb{P}^1\}$. In the terminology of [WJG], this is an *axial moving plane* with L_1 as axis. The second column of A also gives an axial moving plane with axis L_2, the difference being that in (4.3), the axes L_1 and L_2 are disjoint (and the curve is smooth), while in (4.4), the axes meet at $(0,0,0,1)$ (and the curve is singular at this point).

Let us look at (4.3) more closely. Recall that the parametrization (a_0, a_1, a_2, a_3) is given by the signed maximal minors of A. If we set $h_1 = sr_3 - tr_2$ and $h_2 = sr_1 - tr_0$, then one easily computes that

$$a_0 = th_1,\ a_1 = -sh_1,\ a_2 = -th_2,\ a_3 = sh_2. \tag{4.5}$$

Note that h_1 and h_2 have degree $n-1$.

A first consequence of (4.5) is that $a_0a_3 = a_1a_2$, so that the curve lies on the smooth quadric surface $x_0x_3 = x_1x_2$ in $\mathbb{P}^3$. We will see that this surface is a particularly simple example of a *rational normal scroll*. In terms of ideals, (4.5) implies that

$$I = \langle a_0, a_1, a_2, a_3\rangle = \langle th_1, sh_1, th_2, sh_2\rangle = \langle h_1, h_2\rangle \cap \textstyle\bigoplus_{m=n}^{\infty} R_m.$$

Here, recall that $R = \mathsf{k}[s,t]$, so that the above equation tells us that I consists of all elements of degree $m \geq n$ in the simpler ideal $\langle h_1, h_2\rangle$. In the terminology of [I2], $\langle h_1, h_2\rangle$ is the *ancestor ideal* of I.

For (4.4), we have a similar situation. Let (a_0, a_1, a_2, a_3) be the parametrization coming from (4.4) and set $h_1 = r_3$ and $h_2 = a_3$. Taking the first three maximal minors of A, we have

$$a_0 = t^2h_1,\ a_1 = -sth_1,\ a_2 = s^2h_1,\ a_3 = h_2. \tag{4.6}$$

Here h_1 has degree $n-2$ and h_2 has degree n.

From (4.6) we see that $a_0a_2 = a_1^2$, so that the curve lies on the singular quadric surface $x_0x_2 = x_1^2$ in $\mathbb{P}^3$. This surface is another example of a rational normal scroll. In terms of ideals, (4.6) implies that

$$I = \langle a_0, a_1, a_2, a_3\rangle = \langle t^2h_1, sth_1, s^2h_1, h_2\rangle = \langle h_1, h_2\rangle \cap \textstyle\bigoplus_{m=n}^{\infty} R_m.$$

Again, $\langle h_1, h_2\rangle$ is the ancestor ideal of I.

This example shows that the $\boldsymbol{\mu}$-stratum for $(1,1,n-2)$ decomposes into two parts corresponding to (4.3) and (4.4), each of which has a rational normal scroll and an ancestor ideal. There is some rich geometry and algebra going on here.

In general, the *ancestor ideal* of $I = \langle a_0,\dots,a_d\rangle \subseteq R$ is the largest homogeneous ideal of R that equals I in degrees n and higher (remember that $a_0,\dots,a_d \in R_n$). Here is a result from [I2], whose proof we defer until Section A.4.

Theorem 4.9. *Let $I = \langle a_0,\dots,a_d\rangle \subseteq R$ have $\boldsymbol{\mu}$-type $\boldsymbol{\mu}_{\min} = (1,\dots,1,n-d+1)$, $n \geq d+1$. Then the ancestor ideal of I is equal to $\langle h_1, h_2\rangle$, where h_1, h_2 are relatively prime and satisfy*

$$I_n = R_{n-\deg(h_1)} \cdot h_1 \oplus R_{n-\deg(h_2)} \cdot h_2.$$

Since I_n has vector space dimension $d+1$, this theorem implies that

$$d+1 = \alpha_1 + \alpha_2, \quad \alpha_i = n+1-\deg(h_i) \geq 1.$$

If we assume $\deg(h_1) \geq \deg(h_2)$, then $\alpha_1 \leq \alpha_2$, so that we have the partition

$$\mathcal{A} = (\alpha_1, \alpha_2)$$

of $d+1$.

This partition determines the *rational normal scroll* $S_{\alpha_1-1,\alpha_2-1} \subseteq \mathbb{P}^d$, which consists of the points

$$\lambda(s^{\alpha_1-1}, s^{\alpha_1-2}t\dots, t^{\alpha_1-1}, 0, \dots, 0) + \mu(0, \dots, 0, s^{\alpha_2-1}, s^{\alpha_1-2}t\dots, t^{\alpha_2-1})$$

for all $(s,t), (\lambda, \mu)$ in $\mathbb{P}^1$. In this formula, the first expression in parentheses is the *rational normal curve* of degree $\alpha_1 - 1$, sitting in the first α_1 coordinates of $\mathbb{P}^d$; the second expression in parentheses is the rational normal curve of degree α_2-1, sitting in the last α_2 coordinates of $\mathbb{P}^d$. Note how this uses $\alpha_1+\alpha_2 = d+1$. These two rational normal curves are the "edges" of the scroll, which consists of lines parametrized by (λ, μ) that connect the points on the two edges with the same parameter value (s,t). It is well-known that $S_{\alpha_1-1,\alpha_2-1}$ is a surface of degree $d-1$ in $\mathbb{P}^d$ (see [EH]).

In our partition, we assume $1 \leq \alpha_1 \leq \alpha_2$. When $\alpha_1 = 1$, the "rational normal curve of degree 0" is just a point $(1, 0, \dots, 0)$. Since $\alpha_1 + \alpha_2 = d+1$, we have $\alpha_2 = d$, so that the rational normal scroll $S_{\alpha_1-1,\alpha_2-1} = S_{0,d-1}$ is just the cone over the rational normal curve of degree $d-1$ sitting in the last d coordinates of $\mathbb{P}^d$.

To see how this scroll relates to the paramerization given by $I = \langle a_0, \dots, a_d\rangle$, we use Theorem 4.9 to write the ideal as

$$I = \langle s^{\alpha_1-1}h_1, \dots, t^{\alpha_1-1}h_1, s^{\alpha_2-1}h_2, \dots, t^{\alpha_2-1}h_2\rangle.$$

Switching to these generators of I corresponds to a coordinate change in $\mathbb{P}^d$. For $(s,t) \in \mathbb{P}^1$, the parametrization gives the point

$$h_1(s,t)(s^{\alpha_1-1}, \dots, t^{\alpha_1-1}, 0, \dots, 0) + h_2(s,t)(0, \dots, 0, s^{\alpha_2-1}, \dots, t^{\alpha_2-1}), \tag{4.7}$$

which is clearly on $S_{\alpha_1-1,\alpha_2-1}$. Hence we have proved:

Corollary 4.10. *Let $I = \langle a_0, \dots, a_d\rangle \subseteq R$ have $\boldsymbol{\mu}$-type $\boldsymbol{\mu}_{\min} = (1, \dots, 1, n-d+1)$, $n \geq d+1$. If the ancestor ideal of I gives the partition $\mathcal{A} = (\alpha_1, \alpha_2)$ of $d+1$, then after a change of coordinates in $\mathbb{P}^d$, the parametrized curve lies on the rational normal scroll $S_{\alpha_1-1,\alpha_2-1}$.*

Example 4.11. When $d = 3$ and $\boldsymbol{\mu}_{\min} = (1, 1, n-2)$, the only two partitions of 4 are $4 = 2+2 = 3+1$. The corresponding rational normal scrolls are $S_{1,1}$, defined by $x_0x_3 = x_1x_2$, and $S_{0,2}$, defined by $x_1x_3 = x_2^2$. Hence we recover (after a small change of coordinates) the two quadric surfaces encountered in Example 4.8.

Finally, fix a partition $\mathcal{A} = (\alpha_1, \alpha_2)$ of $d+1$ with $1 \leq \alpha_1 \leq \alpha_2$. Then define $\mathcal{P}^{\boldsymbol{\mu}_{\min}}_{n,d,\mathcal{A}} \subseteq \mathcal{P}^{\boldsymbol{\mu}_{\min}}_{n,d}$ to be the subset consisting of *all* parametrizations in the stratum whose ancestor ideal gives the partition $\mathcal{A}$. Recall from Proposition 3.12 that $\dim(\mathcal{P}^{\boldsymbol{\mu}_{\min}}_{n,d}) = d^2 + d + n$ when $n \geq d+1$. We defer the proof of the following result until Section A.4.

Theorem 4.12. *The subsets $\mathcal{P}^{\boldsymbol{\mu}_{\min}}_{n,d,\mathcal{A}} \subseteq \mathcal{P}^{\boldsymbol{\mu}_{\min}}_{n,d}$ have the following properties:*

1. $\mathcal{P}^{\boldsymbol{\mu}_{\min}}_{n,d,\mathcal{A}}$ *is irreducible and is open in its Zariski closure in* $\mathcal{P}^{\boldsymbol{\mu}_{\min}}_{n,d}$
2. $\dim(\mathcal{P}^{\boldsymbol{\mu}_{\min}}_{n,d,\mathcal{A}}) = d^2 + d + 2n - \max(0, a_2 - a_1 - 1)$.
3. *The Zariski closure* $\overline{\mathcal{P}}^{\boldsymbol{\mu}_{\min}}_{n,d,\mathcal{A}}$ *in* $\mathcal{P}^{\boldsymbol{\mu}_{\min}}_{n,d}$ *satisfies*

$$\overline{\mathcal{P}}^{\boldsymbol{\mu}_{\min}}_{n,d,\mathcal{A}} = \bigcup_{\mathcal{A}' \leq \mathcal{A}} \mathcal{P}^{\boldsymbol{\mu}_{\min}}_{n,d,\mathcal{A}'},$$

where the union is over 2*-part partitions* $\mathcal{A}'$ *of* $d+1$.

Example 4.13. Continuing our study of $(1,1,n-2)$ with $n \geq 4$, we have the partitions $(1,3) \leq (2,2)$. The larger partition $(2,2)$ is generic and gives an open dense subset of $\mathcal{P}^{(1,1,n-2)}_{n,3}$, of dimension $2n+12$. Parametrizations with the smaller partition $(1,3)$ lie in a closed subset of codimension 1 since $\max(0, 3-1-1) = 1$.

In the above example, recall from [WJG, Cor. 6.8] that the curves corresponding to $(2,2)$ are smooth while those for $(1,3)$ have a unique singular point of multiplicity $n-2$. It would be interesting to study the singularities of curves in $\mathcal{P}^{\boldsymbol{\mu}_{\min}}_{n,d,\mathcal{A}}$ in the general case when $\boldsymbol{\mu}_{\min} = (1,\dots,1,n-d+1)$.

We conclude by noting that *all* ideals coming from parametrizations, not just those in the smallest stratum, have ancestor ideals that determine rational normal scrolls (possibly of high dimension) containing the curve. More precisely, suppose we have $I = \langle a_0,\dots,a_d\rangle$, where $a_0,\dots,a_d \in R_n$ are linearly independent and relatively prime. If I has $\boldsymbol{\mu}$-type $\boldsymbol{\mu} = (\mu_1,\dots,\mu_d)$, then we will see in Appendix A that the ancestor ideal of I can be written

$$\langle h_1,\dots,h_\tau\rangle,$$

where

$$\tau = d+1-\#\{i \mid \mu_i = 1\} \tag{4.8}$$

and

$$I_n = R_{n-\deg h_1}\cdot h_1 \oplus R_{n-\deg h_2}\cdot h_2 \oplus \cdots \oplus R_{n-\deg h_\tau}\cdot h_\tau. \tag{4.9}$$

Note that $\tau = 2$ precisely when $\boldsymbol{\mu}$ has $d-1$ indices with $\mu_i = 1$, i.e., when $\boldsymbol{\mu} = (1,\dots,1,n-d+1)$ and $n \geq d+1$.

The decomposition (4.9) of I_n gives the partition

$$d+1 = \sum_{i=1}^{\tau} \alpha_i, \quad \alpha_i = n+1-\deg h_i. \tag{4.10}$$

Setting $\mathcal{A} = (\alpha_1,\dots,\alpha_\tau)$ determines a subset $\mathcal{P}^{\boldsymbol{\mu}}_{n,d,\mathcal{A}}$, which as we will see in Section A.4 has codimension and closure properties similar to those stated in Theorem 4.12.

The partition (4.10) gives a rational normal scroll $S_{\alpha_1-1,\dots,\alpha_\tau-1} \subseteq \mathbb{P}^d$. This is formed by putting τ rational normal curves in $\mathbb{P}^d$ using the first α_1 coordinates for the first curve, the next α_2 coordinates for the second, and so on. This works since the α_i partition $d+1$. As in the surface case considered earlier, the "rational normal curve" reduces to a point when $\alpha_i = 1$.

The τ rational normal curves are the "edges" of the scroll. For a fixed parameter value (s,t), we get τ points, one on each of the τ curves. These points determine a subspace of dimension $\tau - 1$. Then $S_{\alpha_1-1,\dots,\alpha_\tau-1}$ is the union of these subspaces as we vary (s,t) over $\mathbb{P}^1$.

When $(a_0,\dots,a_d)$ have no common factor, it is not hard to show that

$$2 \le \tau \le \min\{d+1, n+1-d\}.$$

If $\tau = d+1$, it is easy to see that there is a unique partition $\mathcal{A} = (1,\dots,1)$ and $S_{0,\dots,0}$ is just $\mathbb{P}^d$; and if $\tau = d$, then the unique partition is $\mathcal{A} = (1,\dots,1,2)$ and we again get $\mathbb{P}^d$. So the interesting cases are when $\tau \le d-1$. Here $S_{\alpha_1-1,\dots,\alpha_\tau-1}$ has dimension τ, and one can show that its degree is

$$\deg(S_{a_1-1,\dots,a_\tau-1}) = d+1-\tau. \tag{4.11}$$

See [EH] for more on rational normal scrolls.

When I has ancestor ideal $\langle h_1,\dots,h_\tau\rangle$, (4.7) generalizes to show that the corresponding curve lies on $S_{a_1-1,\dots,a_\tau-1}$. By (4.8) and (4.11), the degree of the scroll equals $\#\{i \mid \mu_i = 1\}$. Thus the number of 1's in the $\boldsymbol{\mu}$-type of the parametrization determines the dimension and degree of the scroll containing the curve. Note also that the interesting case $\tau \le d-1$ occurs only when $\#\{i \mid \mu_i = 1\} \ge 2$. In $\mathbb{P}^3$, these are the $\boldsymbol{\mu}$-types $(1,1,n-2)$ considered in Examples 4.8, 4.11, and 4.13.

Finally, we should mention that the rational normal scrolls discussed here are closely related to (but not the same as) the scrolls considered in [KPU]. They study $\boldsymbol{\mu} = (1,...,1,n-d+1)$, so $\tau = 2$. In this case, our rational normal scroll is the surface $S_{a_1-1,a_2-2} \subseteq \mathbb{P}^d$. In [KPU], they work in $\mathbb{P}^{d+2}$ with homogeneous coordinates $x_0,\dots,x_d,s,t$ and consider the three-dimensional scroll $S_{a_1-1,a_2-2,1} \subseteq \mathbb{P}^{d+2}$.

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A Proofs of the Main Results

Most proofs omitted in Section 3 of the paper can be found in [I2]. However, [I2] is written in the language of Grassmannians, ancestor ideals, and Hilbert functions, so some translation is needed to the situation of this paper. Section A.2 of the appendix is for experts who want to make sure that nothing has been lost in translation. Section A.3 proves the results of Section 4.1; Section A.4 proves the results of Section 4.2 and gives some extensions of those results to non-minimal $\boldsymbol{\mu}$-strata.

A.1 Notation

We have worked with parametrizations $(a_0,\dots,a_d) \in R_n^{d+1}$ and their associated ideals $I = \langle a_0,\dots,a_d\rangle \subseteq R$. In [I2], the focus is on subspaces $V \subseteq R_n$ of dimension $d+1$, and (in the notation of [I2]) the ideal $(V) \subseteq R$ generated by V. One can translate between [I2] and this paper via

$$\begin{aligned} V &\longleftrightarrow \mathrm{Span}(a_0,\dots,a_d) \\ (V) &\longleftrightarrow \langle a_0,\dots,a_d\rangle. \end{aligned}$$

Subspaces $V \subseteq R_n$ of dimension $d+1$ correspond to elements of the Grassmannian $\mathrm{Grass}(d+1, R_n)$. The Hilbert function of the graded k-algebra $R/(V)$ will be denoted H_V. Thus

$$H_V(m) = \dim_{\mathsf{k}}((R/(V))_m)$$

for $m \geq 0$. We say $H_V \leq H_{V'}$ if $H_V(m) \leq H_{V'}(m)$ for all $m \geq 0$.

A.2 Proofs for Section 3

Recall that $\mathcal{CP}_{n,d}$ consists of linearly independent $(d+1)$-tuples $(a_0,\dots,a_d) \in R_n^{d+1}$. This can be regarded as the set of all possible ordered bases of elements of $\mathrm{Grass}(d+1, R_n)$. In particular, we have a map

$$\pi : \mathcal{CP}_{n,d} \longrightarrow \mathrm{Grass}(d+1, R_n)$$

defined by $\pi(a_0,\dots,a_n) = \mathrm{Span}(a_0,\dots,a_d) \subseteq R_n$. We denote by $\mathrm{Grass}^{\boldsymbol{\mu}}_{n,d}$ the image $\pi(\mathcal{P}^{\boldsymbol{\mu}}_{n,d})$, where $\mathcal{P}^{\boldsymbol{\mu}}_{n,d} \subset \mathcal{CP}_{n,d}$ is defined in Section 3.4.

Any two ordered bases of V are related by a unique element of the general linear group $\mathrm{GL}(d+1,\mathsf{k})$. Note also that $\mathrm{GL}(d+1,\mathsf{k})$ is a Zariski open subset of $\mathrm{Mat}_{d+1}(\mathsf{k})$, which is an affine space of dimension $(d+1)^2$. Hence we get the following result that relates parametrizations to subspaces.

Lemma A.1. *The projection* π *makes* $\mathcal{CP}_{n,d}$ *into a locally trivial bundle over* $\mathrm{Grass}(d+1, R_n)$ *with fibre isomorphic to* $\mathrm{GL}(d+1,\mathsf{k})$*. Thus:*

1. *We have an equality of codimensions*

$$\operatorname{codim}(\mathcal{P}^{\boldsymbol{\mu}}_{n,d} \subseteq \mathcal{CP}_{n,d}) = \operatorname{codim}(\operatorname{Grass}^{\boldsymbol{\mu}}_{n,d} \subseteq \operatorname{Grass}(d+1, R_n)).$$

2. *The Zariski closures of* $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ *and* $\operatorname{Grass}^{\boldsymbol{\mu}}_{n,d}$ *are related by*

$$\overline{\mathcal{P}}^{\boldsymbol{\mu}}_{n,d} = \pi^{-1}(\overline{\operatorname{Grass}}^{\boldsymbol{\mu}}_{n,d}).$$

Fix $(d+1)$-dimensional subspaces V and V' of R_n. Since the $\boldsymbol{\mu}$-type depends only on the ideal (see Proposition 3.1), the ideals (V) and (V') have respective $\boldsymbol{\mu}$-types $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$. These ideals also have Hilbert functions H_V and $H_{V'}$. We need the following comparison result.

Lemma A.2. *Suppose V and V' are $(d+1)$-dimensional subspaces of R_n with respective $\boldsymbol{\mu}$-types $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$ and Hilbert functions H_V and $H_{V'}$. Then:*

1. $H_V(m) = n - |\boldsymbol{\mu}|$ *for* $m \gg 0$.
2. $\boldsymbol{\mu}' \leq \boldsymbol{\mu}$ *if and only if* $H_{V'} \geq H_V$.
3. $\boldsymbol{\mu}' = \boldsymbol{\mu}$ *if and only if* $H_{V'} = H_V$.

Proof. We first study H_V. Let $\boldsymbol{\mu} = (\mu_1, \dots, \mu_d)$ with $\mu_i \leq \cdots \leq \mu_d$. By (3.1), the ideal $I = (V) \subseteq R$ gives the free resolution

$$0 \longrightarrow \textstyle\bigoplus_{i=1}^d R(-n-\mu_i) \longrightarrow R(-n)^{d+1} \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

In degree m, this becomes

$$0 \longrightarrow \textstyle\bigoplus_{i=1}^d R_{m-n-\mu_i} \longrightarrow R_{m-n}^{d+1} \longrightarrow R_m \longrightarrow (R/I)_m \longrightarrow 0.$$

Thus

$$\begin{aligned} H_V(m) &= \dim_{\mathsf{k}}((R/I)_m) \\ &= \dim_{\mathsf{k}}(R_m) - (d+1)\dim_{\mathsf{k}}(R_{m-n}) + \sum_{i=1}^d \dim_{\mathsf{k}}(R_{m-n-\mu_i}). \end{aligned} \tag{A.1}$$

Since $\dim_{\mathsf{k}}(R_\ell) = \max(0, \ell+1)$ for all $\ell \in \mathbb{Z}$, an easy computation using (A.1) shows that $H_V(m) = n - |\boldsymbol{\mu}|$ for $m \gg 0$. This proves part (1) of the lemma.

For part (2), set $G_V(m) = \sum_{i=1}^d \dim_{\mathsf{k}}(R_{m-n-\mu_i})$. Since the first two terms in the formula (A.1) for H_V are independent of $\boldsymbol{\mu}$, it follows that

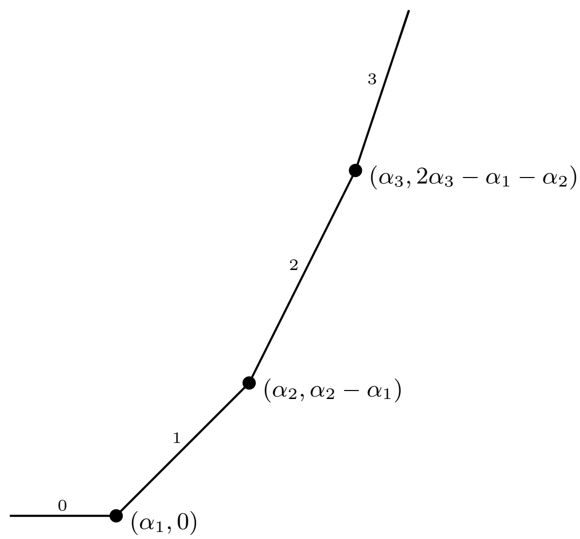
$$G_{V'} \geq G_V \iff H_{V'} \geq H_V.$$

Note that $G_V(m) = \sum_{i=1}^d \max(0, m-n-\mu_i+1)$ since $\dim_{\mathsf{k}}(R_\ell) = \max(0, \ell+1)$. Let $\alpha_i = n + \mu_i - 1$. Then $\alpha_1 \leq \cdots \leq \alpha_d$ and $G_V(m) = \sum_{i=1}^d \max(0, m - \alpha_i)$.

For simplicity, assume that $\alpha_1 < \cdots < \alpha_d$. Then one checks that

$$G_V(\alpha_i) = (\alpha_i - \alpha_1) + \cdots + (\alpha_i - \alpha_{i-1}) = (i-1)\alpha_i - \alpha_1 - \cdots - \alpha_{i-1}.$$

The graph of G_V consists of the points $(\alpha_i, G_V(\alpha_i))$ linked by line segments of slopes $0, 1, 2, \ldots, d$, where the segments of slopes 0 and d are unbounded:



On the interval $\alpha_i \le x \le \alpha_{i+1}$, $G_V(x)$ is linear of slope i and hence is given by

$$G_V(x) = ix - \alpha_1 - \cdots - \alpha_i.$$

Thus the region above the graph is defined by the inequalities

$$y \ge 0,\ y \ge ix - \alpha_1 - \cdots - \alpha_i,\ 1 \le i \le d. \tag{A.2}$$

Now suppose $\boldsymbol{\mu}' = (\mu'_1, \ldots, \mu'_d)$ comes from $V' \subseteq R_n$ and set $\alpha'_i = n + \mu'_i - 1$. Then we have the following equivalences:

$$\begin{aligned} G_{V'} \ge G_V &\iff \text{the graph of } G_{V'} \text{ lies above the graph of } G_V \\ &\iff \text{the graph of } G_{V'} \text{ satisfies the inequalities (A.2)} \\ &\iff (\alpha'_i, (i-1)\alpha'_i - \alpha'_1 - \cdots - \alpha'_{i-1}) \text{ satisfies (A.2) for all } i. \end{aligned}$$

A straightforward computation shows that $(\alpha'_i, (i-1)\alpha'_i - \alpha'_1 - \cdots - \alpha'_{i-1})$ satisfies $y \ge ix - \alpha_1 - \cdots - \alpha_i$ if and only if

$$\alpha'_1 + \cdots + \alpha'_i \le \alpha_1 + \cdots + \alpha_i.$$

This inequality holds for all i when $G_{V'} \ge G_V$. Then $\boldsymbol{\mu}' \le \boldsymbol{\mu}$ follows immediately since $\alpha'_i = n + \mu'_i - 1$ and $\alpha_i = n + \mu_i - 1$. The converse takes more work, since one has to prove that $\boldsymbol{\mu}' \le \boldsymbol{\mu}$ implies that for all i, j, $(\alpha'_i, (i-1)\alpha'_i - \alpha'_1 - \cdots - \alpha'_{i-1})$ satisfies $y \ge jx - \alpha_1 - \cdots - \alpha_j$. We omit the details.

Finally, part (3) of the lemma follows immediately from part (2). □

The graph in the above proof is related to a Harder-Narasimham partial order on the direct sums of line bundles on $\mathbb{P}^1$ (see [I2, Definition 2.26]).

We complete the dictionary between Hilbert functions and $\boldsymbol{\mu}$-types as follows.

Lemma A.3. *Fix n and d with $n \geq d$. Then the map sending H_V to the $\boldsymbol{\mu}$-type $\boldsymbol{\mu}$ of the ideal (V) induces a well-defined bijection between the following two sets:*

1. *The set of Hilbert functions T such that $T = H_V$ for some subspace $V \in \mathrm{Grass}(d+1, R_n)$.*
2. *The set of d-part partitions $\boldsymbol{\mu}$ satisfying $d \leq |\boldsymbol{\mu}| \leq n$.*

It follows that there are only finite many Hilbert functions T in (1).

Remark A.4. We use T to denote a Hilbert function of the form H_V for $V \in \mathrm{Grass}(d+1, R_n)$ in order to match the notation of [I2]. The reason for using T will become clear later in the appendix.

Proof. Lemma A.2(3) implies that $H_V \mapsto \boldsymbol{\mu}$ gives a well-defined injection from (1) to (2). It remains to prove that it is a surjection, i.e., that every d-part partition from (2) is the $\boldsymbol{\mu}$-type of an ideal (V) for some $V \in \mathrm{Grass}(d+1, R_n)$.

Given $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d)$ as in (2), define $T : \mathbb{N} \to \mathbb{N}$ by

$$T(m) = \dim_{\mathsf{k}}(R_m) - (d+1)\dim_{\mathsf{k}}(R_{m-n}) + \sum_{i=1}^{d} \dim_{\mathsf{k}}(R_{m-n-\mu_i}).$$

Using $\dim_{\mathsf{k}}(R_\ell) = \max(0, \ell+1)$, one easily check that H satisfies

$$T(m) = \begin{cases} m+1, & \text{if } 0 \leq m \leq n-1, \\ n-d, & \text{if } m = n, \\ n - |\boldsymbol{\mu}|, & \text{if } m \gg 0. \end{cases}$$

Furthermore, the inequality $\max(0, \ell-1) + \max(0, \ell+1) \geq 2\max(0, \ell)$ makes it easy to show that

$$T(m-1) + T(m+1) \geq 2T(m) \text{ whenever } m \geq n.$$

Setting $e(m) = T(m-1) - T(m)$, it follows that $e(m) \geq e(m+1)$ for all $m \geq n$. By [I1, Proposition 4.6], we conclude that $T = H_V$ for some $V \in \mathrm{Grass}(d+1, R_n)$. This proves the desired surjectivity. $\square$

Given a Hilbert function T as in Lemma A.3(1), we let $\mathrm{GA}_T(d+1, n)$ be the set of all $V \in \mathrm{Grass}(d+1, R_n)$ such that T is the Hilbert function of $R/(V)$ (see [I2, Definition 2.16]). Since there are only finitely many T's, the $\mathrm{GA}_T(d+1,n)$'s partition $\mathrm{Grass}(d+1, R_n)$ into finitely many disjoint sets.

From [I2, Theorems 2.17 and 2.32] we have

Theorem A.5. *Let T be a Hilbert function as in* Lemma A.3(1). *Then:*

1. *$\mathrm{GA}_T(d+1, n)$ is irreducible.*
2. *The Zariski closure $\overline{\mathrm{GA}_T}(d+1, n) = \bigcup_{T' \geq T} \mathrm{GA}_{T'}(d+1, n)$, where the union is over all $T' \geq T$ from* Lemma A.3(1).

We also have the following codimension result from [I2].

Theorem A.6. *Let T be a Hilbert function as in* Lemma A.3(1), *and let $\boldsymbol{\mu}$ be the corresponding d-part partition. Then the codimension of $\mathrm{GA}_T(d+1, n)$ in $\mathrm{Grass}(d+1, R_n)$ is given by the formula*

$$\mathrm{codim}(\mathrm{GA}_T(d+1, n)) = (n - |\boldsymbol{\mu}|)d + \sum_{i>j} \max(0, \mu_i - \mu_j - 1).$$

This follows from [I2, Theorem 2.24 (2.59)] since $n - |\boldsymbol{\mu}| = \lim_{m\to\infty} T(m)$ and the partition D from [I2, Definition 2.21] is just $\boldsymbol{\mu}$ written in descending order.

Proof of Theorems 3.3, 3.7, 3.13 and 3.14. First note that Theorems 3.3 and 3.7 follow from Theorems 3.13 and 3.14 by intersecting with the open set $\mathcal{P}_{n,d} \subseteq \mathcal{CP}_{n,d}$.

The next observation is that if T corresponds to $\boldsymbol{\mu}$ via Lemma A.3, then $\mathrm{Grass}^{\boldsymbol{\mu}}_{n,d}$ from Lemma A.1 is precisely the set $\mathrm{GA}_T(d+1,n)$ since $R/(V)$ has Hilbert function $T = H_V$ if and only if (V) has $\boldsymbol{\mu}$-type $\boldsymbol{\mu}$.

Theorem 3.14 is now an immediate consequence of Theorem A.5 via Lemmas A.1, A.2 and A.3. The irreducibility assertion of Theorem 3.13 follows from Theorem A.5 and Lemma A.1, and the same results imply that

$$\dim(\mathcal{P}^{\boldsymbol{\mu}}_{n,d}) = (d+1)(n+1) - \Big((n-|\boldsymbol{\mu}|)d + \sum_{i>j} \max(0, \mu_i - \mu_j - 1)\Big)$$

since $\dim(\mathcal{P}_{n,d}) = \dim(R_n^{d+1}) = (d+1)(n+1)$. This easily reduces to the formula given in Theorem 3.13.

It remains to show that $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ is open in its Zariski closure. This follows from the disjoint union

$$\overline{\mathcal{P}^{\boldsymbol{\mu}}_{n,d}} = \mathcal{P}^{\boldsymbol{\mu}}_{n,d} \cup \bigcup_{\boldsymbol{\mu}'<\boldsymbol{\mu}} \mathcal{P}^{\boldsymbol{\mu}'}_{n,d}$$

since the large union on the right is easily seen to be closed by Theorem 3.14. □

A.3 Proofs for Section 4.1

We begin with Proposition 4.1.

Proof of Proposition 4.1. Half of the proof was given in Section 4.1. For the other half, assume $n \mid k$ and $n \geq kd$. Then $n - kd + k \geq k$, so that

$$\boldsymbol{\mu} = (\underbrace{k,\ldots,k}_{d-1}, n-kd+k)$$

is a d-part partition of n. Since $\boldsymbol{\mu}$ is divisible by k, Theorem 4.5 implies that $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ contains parametrizations of generic degree k. Thus the same is true for $\mathcal{P}_{n,d}$. □

We next turn to Proposition 4.2.

Proof of Proposition 4.2. First assume $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ contains a parametrization $(a_0,\ldots,a_d)$ of generic degree k. By Lüroth's Theorem (see Section 6.1 of [SWP]), $n = km$, $m \in \mathbb{Z}$, and there are relatively prime $\alpha, \beta \in R_k$ and $b_0,\ldots,b_d \in R_m$ such that

$$a_i(s,t) = b_i(\alpha(s,t),\beta(s,t)), \quad i = 0,\ldots,d. \tag{A.3}$$

([SWP] focuses on the affine case, but their treatment of non-proper parametrizations easily translates to the projective setting used here.)

Note that the b_i are linearly independent and relatively prime since the a_i are (by assumption). Let $\tilde{\boldsymbol{\mu}} = (\tilde{\mu}_1,\ldots,\tilde{\mu}_d)$ be the $\boldsymbol{\mu}$-type of $(b_0,\ldots,b_n)$. Thus $\tilde{\mu}_1 \leq \cdots \leq \tilde{\mu}_d$ and $\tilde{\mu}_1 + \cdots + \tilde{\mu}_d = m$ since the b_i are relatively prime.

Substituting α, β into a $\boldsymbol{\mu}$-basis of $(b_0,\ldots,b_n)$ gives syzygies of $(a_0,\ldots,a_n)$ of degrees $k\tilde{\boldsymbol{\mu}} = (k\tilde{\mu}_1,\ldots,k\tilde{\mu}_d)$. We call these *composed syzygies*. If we can prove that the composed syzygies form a $\boldsymbol{\mu}$-basis of $(a_0, \ldots, a_n)$, then we will get the desired result, namely $\boldsymbol{\mu} = k\tilde{\boldsymbol{\mu}}$.

Let A' be the $(d+1) \times d$ matrix formed by the composed syzygies. Since the maximal minors of a $\boldsymbol{\mu}$-basis of $(b_0, \ldots, b_n)$ give the b_i up to sign, it follows that the maximal minors of A' give the a_i up to sign. Now let A be the $(d+1) \times d$ matrix formed by a $\boldsymbol{\mu}$-basis of $(a_0, \ldots, a_n)$. Its maximal minors also give the a_i up to sign. Expressing each composed syzygy in terms of the $\boldsymbol{\mu}$-basis gives a matrix equation

$$A' = AQ$$

where Q is a $d \times d$ matrix of homogeneous polynomials. Taking maximal minors gives $a_i = a_i \det(Q)$ for all i, so that $\det(Q) = 1$. Hence Q is an invertible matrix of scalars, which proves that the composed syzygies are a $\boldsymbol{\mu}$-basis of $(a_0, \ldots, a_n)$, hence $\boldsymbol{\mu}$ is divisible by k.

To complete the proof, we next assume that $\boldsymbol{\mu}$ is divisible by k. Then $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ contains a parametrization of generic degree k by Theorem 4.5. □

The proof of Theorem 4.5 will require more work.

Proof of Theorem 4.5. First, note that $m \geq d$ is needed for the non-proper locus to be non-empty. Since k divides $\boldsymbol{\mu}$ implies $n = k\tilde{\mu}_1 + ... + k\tilde{\mu}_d$ and each $\tilde{\mu}_i \geq 1$, this implies $n \geq kd$ so that $m = n/k \geq d$.

Fix $d \geq 2$. We will prove the theorem for all $k > 1$ and $n \geq d$ by complete induction on n. The base case $n = d$ is vacuously true since $n = d$ implies $\boldsymbol{\mu} = (1, \ldots, 1)$, which is divisible by no $k > 1$.

Now assume $n > d$ and that the theorem is true for all m with $d \leq m < n$. Take $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ where $\boldsymbol{\mu}$ is a multiple of k and write $\boldsymbol{\mu} = k\tilde{\boldsymbol{\mu}}$, $\tilde{\boldsymbol{\mu}} = (\tilde{\mu}_1, \ldots, \tilde{\mu}_d)$. As noted in the discussion leading up to the theorem, this implies $k \mid n$. Set $m = n/k$ and note that $m = \tilde{\mu}_1 + \cdots + \tilde{\mu}_d \geq d$. Let

$$U = \{(b_0, \ldots, b_d) \in \mathcal{P}^{\tilde{\boldsymbol{\mu}}}_{m,d} \mid (b_0, \ldots, b_d) \text{ is proper}\}.$$

Since $m < n$, our inductive hypothesis implies that U is nonempty, constructible, and Zariski dense in $\mathcal{P}^{\tilde{\boldsymbol{\mu}}}_{m,d}$. In particular, $\dim(U) = \dim(\mathcal{P}^{\tilde{\boldsymbol{\mu}}}_{m,d})$.

Let $W = \{(\alpha, \beta) \in R_k \times R_k \mid \alpha, \beta \text{ are relatively prime}\}$. Composing $(b_0, \ldots, b_d) \in U$ with $(\alpha, \beta) \in W$ gives $(a_0, \ldots, a_d)$ as in (A.3). The a_i have degree $n = km$ and are relatively prime and linearly indepdendent since the b_i are. Furthermore, the argument following (A.3) shows that $(a_0, \ldots, a_d)$ has $\boldsymbol{\mu}$-type $\boldsymbol{\mu} = k\tilde{\boldsymbol{\mu}}$. It follows that composition gives a map

$$U \times W \longrightarrow \mathcal{P}^{\boldsymbol{\mu}}_{n,d}, \tag{A.4}$$

and the proof of Proposition 4.2 shows that the image of this map consists of all generic degree k parametrizations in $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$. It follows easily that this locus is nonempty, constructible, and has irreducible Zariski closure.

To determine the codimension, we need to study the nonempty fibers of (A.4). If a parametrization $(a_0, \ldots, a_d)$ has generic degree k and image curve C, then the function field $\mathsf{k}(C)$ can be identified with $\mathsf{k}(\alpha/\beta)$ for some $(\alpha, \beta) \in V$ (in [SWP], α/β is denoted $R(t)$). Since $\mathsf{k}(\alpha/\beta) = \mathsf{k}(\alpha'/\beta')$ if and only if α/β and α'/β' are related by a linear fractional transform, we see that (α, β) is unique up to the action of $\mathrm{GL}(2, \mathsf{k})$.

Since this group has dimension 4, it follows that the nonempty fibers of (A.4) all have dimension 4. Hence the generic degree k locus in $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ has dimension

$$\dim(U) + \dim(V) - 4 = \dim(\mathcal{P}^{\tilde{\boldsymbol{\mu}}}_{m,d}) + 2(k+1) - 4 = \dim(\mathcal{P}^{\tilde{\boldsymbol{\mu}}}_{m,d}) + 2(k-1).$$

Hence the codimension is

$$\dim(\mathcal{P}^{\boldsymbol{\mu}}_{n,d}) - \dim(\mathcal{P}^{\tilde{\boldsymbol{\mu}}}_{m,d}) - 2(k-1). \tag{A.5}$$

Recall from Theorem 3.3 that

$$\begin{aligned} \dim(\mathcal{P}^{\boldsymbol{\mu}}_{n,d}) &= (d+1)(n+1) - \sum_{i>j} \max(0, k\tilde{\mu}_i - k\tilde{\mu}_j - 1). \\ \dim(\mathcal{P}^{\tilde{\boldsymbol{\mu}}}_{m,d}) &= (d+1)(m+1) - \sum_{i>j} \max(0, \tilde{\mu}_i - \tilde{\mu}_j - 1). \end{aligned} \tag{A.6}$$

The following lemma will help compute the difference $\dim(\mathcal{P}^{\boldsymbol{\mu}}_{n,d}) - \dim(\mathcal{P}^{\tilde{\boldsymbol{\mu}}}_{m,d})$.

Lemma A.7. *Given $\tilde{\boldsymbol{\mu}} = (\tilde{\mu}_1, \ldots, \tilde{\mu}_d)$, let $S(\tilde{\boldsymbol{\mu}}) = \sum_{i>j} \max(0, \tilde{\mu}_i - \tilde{\mu}_j)$. Then:*

1. *$S(\tilde{\boldsymbol{\mu}}) = \sum_{i>j, \tilde{\mu}_i > \tilde{\mu}_j} \tilde{\mu}_i - \tilde{\mu}_j$.*
2. *$\sum_{i>j} \max(0, \tilde{\mu}_i - \tilde{\mu}_j - 1) = S(\tilde{\boldsymbol{\mu}}) - C$, where $C = \#\{(i,j) \mid i > j,\ \tilde{\mu}_i > \tilde{\mu}_j\}$.*
3. *$S(\tilde{\boldsymbol{\mu}}) \leq (m-d)(d-1)$.*

Proof. The proof of (1) is straightforward, and for (2), we similarly get the formula

$$\sum_{i>j} \max(0, \tilde{\mu}_i - \tilde{\mu}_j - 1) = \sum_{i>j, \tilde{\mu}_i > \tilde{\mu}_j} \tilde{\mu}_i - \tilde{\mu}_j - 1.$$

From here, (2) follows easily.

We turn to (3). If $m = d$, then the desired inequality is true since the only possible $\tilde{\boldsymbol{\mu}}$ is $(1, \ldots, 1)$, for which $S = 0$. Hence we may assume $m > d$. Now write $\tilde{\boldsymbol{\mu}} = (1, \ldots, 1, \tilde{\mu}_{i_0}, \ldots, \tilde{\mu}_d)$, where $\tilde{\mu}_{i_0} > 1$. Let $\tilde{\boldsymbol{\mu}}' = (1, \ldots, \tilde{\mu}_{i_0} - 1, \ldots, \tilde{\mu}_d + 1)$. If we can show that $S(\tilde{\boldsymbol{\mu}}) \leq S(\tilde{\boldsymbol{\mu}}')$, then it will follow that

$$S(\tilde{\boldsymbol{\mu}}) \leq S(1, \ldots, 1, m-d+1) = \sum_{d>j} (m-d+1) - 1 = (m-d)(d-1),$$

and the lemma will be proved.

When we compare $S(\tilde{\boldsymbol{\mu}})$ and $S(\tilde{\boldsymbol{\mu}}')$, we only need to consider pairs $i > j$ where $i = d$ or $i = i_0$ or $j = i_0$ (note $i > j$ implies $j \neq d$). We analyze these as follows:

- For terms with $i = d$, we have increased $\tilde{\mu}_d$ by 1. Since $\tilde{\mu}_d + 1$ is guaranteed to be bigger than every other entry, this increases $\tilde{S(\boldsymbol{\mu}')}$ by $d-1$.
- For terms with $j = i_0$, we have decreased $\tilde{\mu}_{i_0}$ by 1 and since we are subtracting, these terms increase $S(\tilde{\boldsymbol{\mu}}')$.
- For terms with $i = i_0$, the possible j's are $1, \ldots, i_0 - 1$, and since we have decreased $\tilde{\mu}_{i_0}$ by 1, we decrease $S(\tilde{\boldsymbol{\mu}}')$ by $i_0 - 1$.

Since $i_0 \leq d$, the increase offsets the decrease, and $S(\tilde{\boldsymbol{\mu}}) \leq S(\tilde{\boldsymbol{\mu}}')$ follows. $\square$

Completion of Proof of Theorem 4.5. By (A.6) and Lemma A.7, it follows that

$$\dim(\mathcal{P}^{\boldsymbol{\mu}}_{n,d}) = (d+1)(km+1) - (kS - C)$$
$$\dim(\mathcal{P}^{\tilde{\boldsymbol{\mu}}}_{m,d}) = (d+1)(m+1) - (S - C)$$

since $n = km$ and $S = \sum_{i>j} \max(0, \tilde{\mu}_i - \tilde{\mu}_j) = \sum_{i>j, \tilde{\mu}_i > \tilde{\mu}_j} \tilde{\mu}_i - \tilde{\mu}_j$. Combining this with (A.5), we see that the codimension is

$$\begin{aligned} &\dim(\mathcal{P}^{\boldsymbol{\mu}}_{n,d}) - \dim(\mathcal{P}^{\tilde{\boldsymbol{\mu}}}_{m,d}) - 2(k-1) \\ &= (d+1)(km+1) - (kS - C) - \big((d+1)(m+1) - (S - C)\big) - 2(k-1) \\ &= (k-1)(m(d+1) - S - 2). \end{aligned}$$

This proves the desired formula for the codimension.

Since $S \leq (m-d)(d-1)$ by Lemma A.7, it follows that

$$m(d+1) - S - 2 \geq m(d+1) - (m-d)(d-1) - 2 = d(d-1) + 2m - 2.$$

This easily gives the lower bound $(k-1)(d(d-1)+2m-2)$ stated in (1) of the theorem. Furthermore, $d(d-1) + 2m - 2 > 0$ since $m \geq d \geq 2$. It follows that the codimension is always positive, completing the proof of (1).

For (2), we consider all $k > 1$ that divide $\boldsymbol{\mu}$. For any such k, the locus of generic degree k parametrizations has positive codimension. Since there are only finitely many such k's, the same is true for the non-proper locus. Hence $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ contains a nonempty Zariski open subset consisting of proper parametrizations. This subset is dense since $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ is irreducible, and (2) follows.

When $d = 1$, we have $S = C = 0$ and the codimension formula readily follows from (A.5) and (A.6). □

A.4 Proofs for Section 4.2

Ancestor ideals, not mentioned so far in this appendix, play a central role in [I2]. As in Section 4.2, an ideal $I \subseteq R$ generated by elements of degree n has an *ancestor ideal*, which is the largest homogeneous ideal of R that agrees with I in degrees $m \geq n$. When $I = (V)$ for $V \in \mathrm{Grass}(d+1, R_n)$, we will follow [I2] and denote its ancestor ideal by $\overline{V}$.

We denote the Hilbert function of $R/\overline{V}$ by $H_{\overline{V}}$. These Hilbert functions are characterized in [I2, Theorem 2.19]. Given such a function H, let $\mathrm{Grass}_H(d+1, n)$ consist of all $V \in \mathrm{Grass}(d+1, R_n)$ such that $H = H_{\overline{V}}$ (see [I2, Definition 1.11]).

We define the partial order $\geq_{\mathcal{P}}$ on these Hilbert functions by setting $H' \geq_{\mathcal{P}} H$ if and if $H'(m) \geq H(m)$ for $m \geq n$ and $H'(m) \leq H(m)$ for $0 \leq m \leq n$. Then we have [I2, Theorem 2.32]:

Theorem A.8. $\mathrm{Grass}_H(d+1, n)$ *is irreducible and open dense in its Zariski closure, which is given by*

$$\overline{\mathrm{Grass}_H}(d+1, n) = \bigcup_{H' \geq_{\mathcal{P}} H} \mathrm{Grass}_{H'}(d+1, n).$$

For the rest of the appendix, we will work in the relatively prime case. Thus all partitions $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d)$ that appear will be partitions of n, i.e., $|\boldsymbol{\mu}| = n$.

Given $V \in \mathrm{Grass}(d+1, R_n)$, let $h_1, \ldots, h_\tau$ be minimal generators of the ancestor ideal $\overline{V}$. We assume $\deg(h_1) \geq \cdots \geq \deg(h_\tau)$. Note also that $\deg(h_i) \leq n$ for all i since (V) and $\overline{V}$ are equal in degrees $\geq n$. Then [I2, (2.43)] implies the following.

Lemma A.9. *Suppose* $V \in \mathrm{Grass}(d+1, n)$ *has* $\boldsymbol{\mu}$*-type* $\boldsymbol{\mu} = (\mu_1, \dots, \mu_d)$ *with* $|\boldsymbol{\mu}| = n$ *and ancestor ideal* $\overline{V} = \langle h_1, \dots, h_\tau \rangle$ *as above. Then* $\tau = d + 1 - \#\{i \mid \mu_i = 1\}$ *and we have a minimal free resolution*

$$0 \longrightarrow \textstyle\bigoplus_{i=1}^{d} R(-n-\mu_i) \longrightarrow \bigoplus_{i=1}^{\tau} R(-\deg(h_i)) \longrightarrow R \longrightarrow R/\overline{V} \longrightarrow 0.$$

This proposition implies in particular that

$$V = \bigoplus_{i=1}^{\tau} R_{n-\deg(h_i)} \cdot h_i, \tag{A.7}$$

so that if we set $\alpha_i = n + 1 - \deg(h_i)$, then $\alpha_1 + \cdots + \alpha_\tau = d + 1$ and $\alpha_1 \le \cdots \le \alpha_\tau$ since $\deg(h_1) \ge \cdots \ge \deg(h_\tau)$. Thus $\mathcal{A} = (\alpha_1, \dots, \alpha_\tau)$ is a τ-part partition of $d + 1$. Note also that $\boldsymbol{\mu}$ determines the length of $\mathcal{A}$ since $\tau = d + 1 - \#\{i \mid \mu_i = 1\}$ by Lemma A.9.

It follows that V gives two partitions, $\boldsymbol{\mu}$ and $\mathcal{A}$. These partitions have a strong relation to the Hilbert function of the ancestor ideal as follows.

Lemma A.10. *Suppose* $V \in \mathrm{Grass}(d+1, R_n)$ *has partitions* $\boldsymbol{\mu}$ *and* $\mathcal{A}$*, with* $|\boldsymbol{\mu}| = n$*, and Hilbert function* $H = H_{\overline{V}}$ *of* $R/\overline{V}$*. Given another* $V' \in \mathrm{Grass}(d+1, R_n)$ *with partitions* $\boldsymbol{\mu}'$ *and* $\mathcal{A}'$*, such that* $|\boldsymbol{\mu}'| = n$*, and* $H' = H_{\overline{V}'}$*, then*

$$H' \ge_{\mathcal{P}} H \iff \boldsymbol{\mu}' \le \boldsymbol{\mu} \text{ and } \mathcal{A}' \le \mathcal{A}.$$

Proof. First suppose $H' \ge_{\mathcal{P}} H$. Since the ideals (V) and $\overline{V}$ are equal in degrees $\ge n$, it follows that $H_V(m) = H_{\overline{V}}(m)$ for $m \ge n$. The same is true for V'. Since $H_V(m) = H_{V'}(m) = m + 1$ for $0 \le m \le n - 1$, our assumption $H' \ge_{\mathcal{P}} H$ implies that $H_{V'} \ge H_V$, and then $\boldsymbol{\mu}' \le \boldsymbol{\mu}$ follows from Lemma A.2.

From Lemma A.9, we see that for $m \le n$,

$$\begin{aligned} H(m) &= \dim_{\mathsf{k}}(R_m) - \sum_{i=1}^{\tau} \dim_{\mathsf{k}}(R_{m-\deg(h_i)}) \\ &= m + 1 - \sum_{i=1}^{\tau} \max(0, m - \deg(h_i) + 1). \end{aligned}$$

We write this as $H(m) = m+1-G(m)$, where $G(m) = \sum_{i=1}^{\tau} \max(0, m-\deg(h_i)+1)$, and similarly $H'(m) = m+1-G'(m)$, where $G'(m) = \sum_{i=1}^{\tau} \max(0, m-\deg(h_i')+1)$.

Then $H' \ge_{\mathcal{P}} H$ implies $H'(m) \le H(m)$ for $m \le n$, so that $G'(m) \ge G(m)$ for the same m. The proof of Lemma A.2 then implies that

$$(\deg(h_\tau'), \dots, \deg(h_1')) \le (\deg(h_\tau), \dots, \deg(h_1)) \tag{A.8}$$

since $\deg(h_\tau) \le \cdots \le \deg(h_1)$, similarly for $\deg(h_i')$. Using $\alpha_i = n + 1 - \deg(h_i)$ and $\sum_{i=1}^{\tau} \alpha_i = d + 1$, one sees that

$$\deg(h_\tau) + \cdots + \deg(h_{j+1}) = \alpha_1 + \cdots + \alpha_j - (d+1) + (\tau - j)(n+1).$$

It follows that (A.8) is equivalent to $\mathcal{A}' = (\alpha_1', \dots, \alpha_\tau') \le \mathcal{A} = (\alpha_1, \dots, \alpha_\tau)$.

We omit the proof of the other implication. □

Since the Hilbert function H_V of $R/(V)$ equals the Hilbert function $H_{\overline{V}}$ of $R/\overline{V}$ for $m \ge n$, we call $T = H_V$ the *tail* of $H = H_{\overline{V}}$ (see [I2, Definition 2.16]). This explains why we used T for the Hilbert functions occurring earlier in the appendix.

The final result we need is a consequence of [I2, Theorem 2.24].

Theorem A.11. *Let H be associated to partitions $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d)$ and $\mathcal{A} = (a_1, \ldots, a_\tau)$ as above, and assume $|\boldsymbol{\mu}| = n$. Then the codimension of* $\mathrm{Grass}_H(d+1, n))$ *in* $\mathrm{Grass}(d+1, n)$ *satisfies*

$$\operatorname{codim}(\mathrm{Grass}_H(d+1,n)) = \sum_{i>j} \max(0, \mu_i - \mu_j - 1) + \sum_{i>j} \max(0, a_i - a_j - 1).$$

Proof. Recall that the the partition D of [I2, Theorem 2.24] is our μ written in descending order, and A there is our $\mathcal{A}$ written in descending order. The equation (2.61) of Theorem 2.24 there can be rewritten when $c_H = 0$ (no common factor of $(a_1, \ldots, a_n)$) as $\operatorname{codim}(\mathrm{Grass}_H(d+1,n)) = \ell(A) + \ell(D)$, which is the expression above. □

Proofs of Theorems 4.9 and 4.12. Set $\boldsymbol{\mu} = \boldsymbol{\mu}_{\min} = (1, \ldots, 1, n-d+1)$ and note that $\tau = (d+1) - (d-1) = 2$ since $n \geq d+1$. Then Theorem 4.9 follows immediately from Lemma A.9 and (A.7).

Next observe that the set $\mathcal{P}^{\boldsymbol{\mu}}_{n,d,\mathcal{A}}$ from Theorem 4.12 satisfies

$$\mathcal{P}^{\boldsymbol{\mu}}_{n,d,\mathcal{A}} = \pi^{-1}(\mathrm{Grass}_H(d+1,n)), \tag{A.9}$$

where π is from Lemma A.1 and H is the ancestor Hilbert function corresponding to partitions $\boldsymbol{\mu}$ and $\mathcal{A}$. Then the codimension formulas from Theorems A.6 and A.11, together with Lemma A.1, show that $\mathcal{P}^{\boldsymbol{\mu}}_{n,d,\mathcal{A}}$ has codimension $\max(0, a_2 - a_1 - 1)$ in $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ (remember that $|\boldsymbol{\mu}| = n$).

Finally, we compute the Zariski closure of $\mathcal{P}^{\boldsymbol{\mu}}_{n,d,\mathcal{A}}$ in $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$. Applying π^{-1} to Theorem A.8 and intersecting with $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$, we obtain

$$\overline{\mathcal{P}}^{\boldsymbol{\mu}}_{n,d,\mathcal{A}} = \Big(\bigcup_{\boldsymbol{\mu}' \leq \boldsymbol{\mu}, \mathcal{A}' \leq \mathcal{A}} \mathcal{P}^{\boldsymbol{\mu}'}_{n,d,\mathcal{A}'} \Big) \cap \mathcal{P}^{\boldsymbol{\mu}}_{n,d}$$

by (A.9) and Lemmas A.1 and A.10. Since $\mathcal{P}^{\boldsymbol{\mu}'}_{n,d,\mathcal{A}'} \subseteq \mathcal{P}^{\boldsymbol{\mu}'}_{n,d}$ is disjoint from $\mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ for $\boldsymbol{\mu}' \neq \boldsymbol{\mu}$, the expression on the right reduces to the formula in Theorem 4.12. □

We note that $\mathcal{P}^{\boldsymbol{\mu}}_{n,d,\mathcal{A}} \subseteq \mathcal{P}^{\boldsymbol{\mu}}_{n,d}$ can be defined for any d-part partition $\boldsymbol{\mu}$ of n and any τ-part partition $\mathcal{A}$ of $d+1$, where $\tau = d + 1 - \#\{i \mid \mu_i = 1\}$ as in Lemma A.9. Theorems 4.9 and 4.12 easily generalize to this case using the above results from [I2]. Furthermore, if $\mathcal{A} = (\alpha_1, \ldots, \alpha_\tau)$ and $\tau \leq d$, then parametrizations in $\mathcal{P}^{\boldsymbol{\mu}}_{n,d,\mathcal{A}}$ give curves that lie on the τ-dimensional rational normal scroll $S_{\alpha_1-1,\ldots,\alpha_\tau-1} \subseteq \mathbb{P}^d$.